Math 2250-1
Friday Sept. 26

Wed: Matrix ops

→ begin on page 3 Wed., identity matrices, and matrix inverses.

HW for Oct 3

3.5 5, 7, 8, 13, 22, 23, 30, 31, 32, 33.
hand in also techology checks (Maple, Matlab etc): 22: A^{-1}, and that $AA^{-1} = A^{-1}A = I$, 23

3.6 2, 6, 17, 21, 22, 32
in 22 and 32 find x (x_1 in 32) with Cramer's rule. Then compute A^{-1} with the adjoint formula (page 210), as well as with the rref algorithm (better get same answer!) Finally, use A^{-1} to solve $A\vec{x} = \vec{b}$ and compare your Cramer's rule computations to the first components of your sol'ns.

4.1 1, 7, 9, 10, 15, 16, 19, 22, 25, 26, 31, 33

Then, What good is A^{-1}?

Theorem If A^{-1} exists, then the only solution to $A\vec{x} = \vec{b}$ is
$$\vec{x} = A^{-1}\vec{b}$$

Example continued:

Solve $\begin{array}{l} x + 2y = 5 \\ 3x + 4y = 6 \end{array}$ $\longrightarrow \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix} \xrightarrow{\text{Theorem}} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}\begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} -4 \\ 9/2 \end{bmatrix}$

check ans:

check theorem:

But how did I know A^{-1} in that example?
Ans: I solved the matrix eqtn

$AX = Id.$

for $col_1(X)$: $\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 3 & 4 & 0 \end{array}\right]$

for $col_2(X)$: $\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 3 & 4 & 1 \end{array}\right]$

for both columns at once:

$$\begin{array}{r} \\ \\ \\ -3R_1+R_2 \\ R_1+R_2 \\ \\ \end{array}\begin{array}{cc|c|c} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \\ \hline 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \\ \hline 1 & 0 & -2 & 1 \\ 0 & -2 & -3 & 1 \end{array}$$

$$-\frac{R_2}{2}\quad \begin{array}{cc|c|c} 1 & 0 & -2 & 1 \\ 0 & 1 & 3/2 & -1/2 \end{array}$$

$\uparrow col_1 \quad \uparrow col_2$

so $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

This is a general algorithm!

The matrix equation for $X = A^{-1}$ is

$$A X = I \qquad A_{n\times n},\ X_{n\times n},\ I_{n\times n}$$

Synthetically this is the n by $2n$ augmented matrix

$$A \,\vdots\, I$$

(where you are solving for all n columns at once!)

<u>If</u> this reduces to rref equal to

$$I \,\vdots\, B$$

Then $X = B$, so $AB = I$

[notice if you write $B \,\vdots\, I$ and do the row ops backwards you get $I \,\vdots\, A$, so also $BA = I$ is automatic].

so

<u>Thm</u> if $\operatorname{rref}(A) = I$ then A^{-1} exists and can be found by algorithm above.

<u>Thm</u> if A^{-1} exists then unique sol'n to $A\vec{x} = \vec{b}$ is $\vec{x} = A^{-1}\vec{b}$

so, if A^{-1} exists we must also have $\operatorname{rref}(A) = I$! ($\operatorname{rref}(A) \neq I \Rightarrow$ never get unique sol'ns)

$$\boxed{A^{-1} \text{ exists iff } \operatorname{rref}(A) = \text{id. iff } A\vec{x} = \vec{b} \text{ always has unique sol'n}}$$

<u>example</u> $A = \begin{bmatrix} 1 & 5 & 1 \\ 2 & 5 & 0 \\ 2 & 7 & 1 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 5 & 1 & 1 & 0 & 0 \\ 2 & 5 & 0 & 0 & 1 & 0 \\ 2 & 7 & 1 & 0 & 0 & 1 \end{array}\right]$$

$-2R_1+R_2$, $-2R_1+R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 5 & 1 & 1 & 0 & 0 \\ 0 & -5 & -2 & -2 & 1 & 0 \\ 0 & -3 & -1 & -2 & 0 & 1 \end{array}\right]$$

R_2+R_1, $-R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & -1 & 1 & 0 \\ 0 & -5 & -2 & -2 & 1 & 0 \\ 0 & 3 & 1 & 2 & 0 & -1 \end{array}\right]$$

$2R_3+R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 2 & 1 & -2 \\ 0 & 3 & 1 & 2 & 0 & -1 \end{array}\right]$$

$-3R_2+R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 2 & 1 & -2 \\ 0 & 0 & 1 & -4 & -3 & 5 \end{array}\right]$$

R_3+R_1

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & -2 & 5 \\ 0 & 1 & 0 & 2 & 1 & -2 \\ 0 & 0 & 1 & -4 & -3 & 5 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -5 & -2 & 5 \\ 2 & 1 & -2 \\ -4 & -3 & 5 \end{bmatrix}$$

Nice formula for inverse of 2×2 matrix:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} \text{ exists iff } D = ad-bc \neq 0,$$

and in this case, it equals

$$\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}. \qquad \text{(See example 1)}$$

how did we find this?

$ad-bc \neq 0$ is exactly the condition that row_1 and row_2 are not multiples, i.e. $\text{rref}(A) = I$.

do row ops with letters, or check the formula by computing AA^{-1}

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right]$$

$$\begin{array}{l} \frac{1}{a}R_1 \\ \\ \end{array} \left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ c & d & 0 & 1 \end{array}\right] \qquad [\text{if } a \neq 0]$$

$$\begin{array}{l} \\ -cR_1+R_2 \end{array} \left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 0 & d-\frac{cb}{a} & -\frac{c}{a} & 1 \end{array}\right]$$

$$\begin{array}{l} \\ \text{common denom} \end{array} \left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 0 & \frac{D}{a} & -\frac{c}{a} & 1 \end{array}\right]$$

$$\begin{array}{l} \\ \frac{a}{D}R_2 \end{array} \left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 0 & 1 & -\frac{c}{D} & \frac{a}{D} \end{array}\right] \qquad [\text{if } D \neq 0]$$

$$\begin{array}{l} -\frac{b}{a}R_2+R_1 \\ \\ \end{array} \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{a}+\frac{bc}{aD} & -b/D \\ 0 & 1 & -c/D & a/D \end{array}\right] \qquad \frac{D+bc}{aD} = \frac{d}{D}$$

$$\left[\begin{array}{cc|cc} 1 & 0 & \frac{d}{D} & -\frac{b}{D} \\ 0 & 1 & -\frac{c}{D} & \frac{a}{D} \end{array}\right]$$

$$A^{-1} = \frac{1}{D}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \; !$$

(of course, we could've just checked that this magic formula works, but by deriving it we see it's not "magic")