

Math 2250-1  
Monday Sept 22

①

- On Friday we used Gaussian elimination to find all sol'ns to

$$\begin{aligned} x_1 - 2x_2 + 3x_3 + 2x_4 + x_5 &= 10 \\ 2x_1 - 4x_2 + 8x_3 + 3x_4 + 10x_5 &= 7 \\ 3x_1 - 6x_2 + 10x_3 + 6x_4 + 5x_5 &= 27 \end{aligned}$$

synthetic version

$$\begin{array}{ccccc|c} 1 & -2 & 3 & 2 & 1 & 10 \\ 2 & -4 & 8 & 3 & 10 & 7 \\ 3 & -6 & 10 & 6 & 5 & 27 \end{array}$$

Gaussian elimination, stage 1 (working to the right and down)

$$\begin{array}{ccccc|c} 1 & -2 & 3 & 2 & 1 & 10 \\ 0 & 0 & -2 & -1 & 8 & -3 \\ 0 & 0 & 4 & 2 & 2 & 7 \end{array}$$

- row echelon form (not unique, i.e. a given matrix can have different row echelon forms)
- all zero rows at bottom
  - leading non-zero entry in each row is strictly to the right of the previous rows

Gaussian elimination, stage 2 (working up and to the left!)

$$\begin{array}{ccccc|c} 1 & -2 & 0 & 0 & 3 & 5 \\ 0 & 0 & 1 & 0 & 2 & -3 \\ 0 & 0 & 0 & 1 & -4 & 7 \end{array}$$

stands for

$$\begin{aligned} x_1 - 2x_2 + 3x_3 + 2x_4 + x_5 &= 10 \\ x_3 + 2x_5 &= -3 \\ x_4 - 4x_5 &= 7 \end{aligned}$$

backsolve:

$$\begin{aligned} x_5 &= t \\ x_4 &= 7 + 4t \\ x_3 &= -3 - 2t \\ x_2 &= s \\ x_1 &= 10 - t - 2(-3 - 2t) - 3(-3 - 2t) + 2s \\ &= 5 - 3t + 2s \end{aligned}$$

reduced row echelon form (unique!!)

- ①, ②, and
- each non-zero row leads with a 1, called the leading 1
- each column with one of the row leading 1's, has zeroes in every other column entry.

stands for

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= 5 \\ x_3 + 2x_5 &= -3 \\ x_4 - 4x_5 &= 7 \end{aligned}$$

backsolve

$$\begin{aligned} x_5 &= t \\ x_4 &= 7 + 4t \\ x_3 &= -3 - 2t \\ x_2 &= s \\ x_1 &= 5 - 3t + 2s \end{aligned}$$

in vector form:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ -3 \\ 7 \\ 0 \end{bmatrix} + s \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ -2 \\ 4 \\ 1 \end{bmatrix}; s, t \in \mathbb{R}$$

Math 2250-1  
September 22, 2008  
Maple linear algebra

```
[ > with(linalg): #load linear algebra package
Coefficient matrix taken from problem #19, section 3.3, page 170. (You are assigned #20).
> A:= matrix(3,5,[2,7,-10,-19,13,1,3,-4,-8,6,1,0,2,1,3]);
#the coefficient matrix
```

$$A := \begin{bmatrix} 2 & 7 & -10 & -19 & 13 \\ 1 & 3 & -4 & -8 & 6 \\ 1 & 0 & 2 & 1 & 3 \end{bmatrix}$$

```
> rref(A); #compute the reduced row echelon form
```

$$\begin{bmatrix} 1 & 0 & 2 & 1 & 3 \\ 0 & 1 & -2 & -3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

I want to consider three different linear systems for which A is the coefficient matrix. In the first one, the right hand sides are all zero (what we call the "homogeneous" problem), and I have carefully picked the other two right hand sides (by working backwards from rref(A), actually). The three right hand sides are the columns of the matrix B:

```
> B:=matrix(3,3,[0,7,7,0,0,3,0,0,0]);
```

$$B := \begin{bmatrix} 0 & 7 & 7 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

We can consider all three linear systems at once by augmenting B to A, and then rref-ing: (you'll have to put in the vertical line separating the coefficient matrix from the right hand sides by yourself.)

```
> C:=augment(A,B);
rrefC:=rref(C);
```

$$C := \begin{bmatrix} 2 & 7 & -10 & -19 & 13 & | & 0 & 7 & 7 \\ 1 & 3 & -4 & -8 & 6 & | & 0 & 0 & 3 \\ 1 & 0 & 2 & 1 & 3 & | & 0 & 0 & 0 \end{bmatrix}$$
  
$$rrefC := \begin{bmatrix} 1 & 0 & 2 & 1 & 3 & | & 0 & 0 & 0 \\ 0 & 1 & -2 & -3 & 1 & | & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & | & 0 & 1 & 0 \end{bmatrix}$$

From rref(C) you can read off the solutions to each of the three problems. Note that each column (from rref(A)) without a leading 1 in it gives you a free parameter in the solution. Write down the solutions to our three problems on the next page:

```
> rref(C);
```

$$\left[ \begin{array}{cccccc|ccc} 1 & 0 & 2 & 1 & 3 & 0 & 0 & 0 \\ 0 & 1 & -2 & -3 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array} \right]$$

**Important conceptual questions:**

(1) Which of these three solutions could you have written down just from rref(A), rather than from the rref of the augmented matrix? Why?

(2) Linear systems in which right hand side vector equals zero are called homogeneous linear systems. Otherwise they are called inhomogeneous. Notice that the general solution to the consistent inhomogeneous system is the sum of a particular solution to it, together with the general solution to the homogeneous system!!! Was this an accident? We'll come back to this.

By the way, Maple will go ahead and solve a linear system directly if you ask it to. Here's the consistent inhomogeneous problem from the previous example:

```
[ > b:=vector([7,3,0]);
                                     b := [7,3,0]
[ > linsolve(A,b);
                                     [-2 _t1 - _t2 - 3 _t3, 2 _t1 + 3 _t2 - _t3 + 1, _t1, _t2, _t3]
```

Is this the answer we got?

### Using $\text{rref}(A)$ to discern general facts about the solutions to $Ax=b$ :

The reduced row echelon form of a matrix tells you a lot about possible solutions to the matrix equation  $Ax=b$ . What can you say in the following situations?

```
[ > AA:=matrix(2,5,[2, 7, -10, -19, 13,1, 3, -4, -8, 6]);
  rref(AA);
                                     AA := [ 2  7 -10 -19 13 ]
                                               [ 1  3  -4  -8  6 ]
                                               [ 1  0  2  1  3 ]
                                               [ 0  1 -2  -3  1 ]
```

Is the homogeneous problem  $Ax=0$  always solvable?

Is the inhomogeneous problem  $Ax=b$  always solvable?

When the inhomogeneous (or homogeneous) problem is solvable, how many solutions are there?

```

> B:=matrix(3,2,[1,2,-1,3,4,2]);
      B :=  $\begin{bmatrix} 1 & 2 \\ -1 & 3 \\ 4 & 2 \end{bmatrix}$ 
> RREFB:=rref(B);
      RREFB :=  $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ 

```

How many solutions to the homogeneous problem  $Bx=0$ ?

Is the inhomogeneous problem  $Bx=b$  always solvable?

When it is solvable, how many solutions does it have?

```

> C:=matrix(4,4,[1,0,-1,1,22,-1,3,5,7,4,6,2,3,5,7,13]);
      C :=  $\begin{bmatrix} 1 & 0 & -1 & 1 \\ 22 & -1 & 3 & 5 \\ 7 & 4 & 6 & 2 \\ 3 & 5 & 7 & 13 \end{bmatrix}$ 
> RREFC:=rref(C);
      RREFC :=  $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ 

```

Square matrices with 1's down the diagonal (which runs from the upper left to lower right corner) are special. They are called identity matrices.

How many solutions to the homogeneous problem  $Cx=0$ ?

Is the inhomogeneous problem  $Cx=b$  always solvable?

How many solutions?

What are your general conclusions?

(1) What conditions on  $\text{rref}(A)$  guarantee that the homogeneous equation  $Ax=0$  has infinitely many solutions?

(2) What conditions on the dimensions of  $A$  always force infinitely many solutions to the homogeneous problem regardless of the individual entries of  $A$ ?

(3) What conditions on  $\text{rref}(A)$  guarantee that solutions  $x$  to  $Ax=b$  are always unique (if they exist)?

(4) If  $A$  is a square matrix, what can you say about solutions to  $Ax=b$  when  
(4a)  $\text{rref}(A)$  is the identity matrix

(4b)  $\text{rref}(A)$  is not the identity matrix?