Math 2250-1
Newton's law of cooling application - Maple Project 1
Notes and assignment
Friday September 5

Our first Maple project is from section 1.5, First Order Linear Differential Equations, specifically from *1.5 Application: Indoor Temperature Oscillations*, pages 56-58. We have adapted Professor Gustafson's version of this project which he wrote for his Math 2250 section. I recommend you work on this project with your text open. You may also want to consult the Maple hints on Professor Gustafson's page. (Even though our questions may not look exactly the same as his, they are related.) The Maple Introduction notes I recommended to you last week may also be helpful, especially if you're a Maple beginner. Links to all this material are on our Maple home page, which is linked from our course home page, http://math.utah.edu/~korevaar/2250fall08

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Project Directions: You will be handing in a single Maple document containing the answers to the bolded, underlined exercises below. At the top of this document you should have a text field with an appropriate title, the date, your name and uid. Format this information in a pleasing way. Below the title/name/date header you should answer the **exercises** in order. The exercises which call for hand work (e.g. #1), can be done by leaving enough text space in your document so that you can write in the relevant explanations/derivations by hand. Alternately, you may create text displays of your "hand" computations, as is done in this assignment document. To follow this second, harder but neater alternative, past Maple output into text fields, as I've done below. (Then delete the Maple fields.)
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Here's the project set-up as explained in the text, page 56: You have no air conditioner or evaporative cooler for your home, it's summer, you live in the authors' home town of Athens Georgia, and the outside temperatures are oscillating periodically, with a period of 24 hours. In other words, we will assume the ambient temperature outside is given by

$$A(t) = a_0 + a_1 \cos(\omega t) + b_1 \sin(\omega t)$$

where the angular frequency ω is 2π radians per 24 hours, i.e.

$$\omega = \frac{\pi}{12}$$

radians per hour. In the book example the 24-hour average temperature is 80 degrees F, with a minimum of 70 degrees at 4 a.m., and a maximum of 90 degrees at 4 p.m. This is consistent with the fact that the hottest part of the day usually occurs in the afternoon, and the coolest part is in the early morning hours. Using this information, we may write

$$A(t) = 80 - 10 \cos(\omega (t - 4)) \qquad\qquad (1)$$

and expand using the cosine addition angle formula, which you should have memorized and which is in your Calculus book:

$$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

This yields

$$A(t) = 80 - 5\cos(\omega t) - 5\sqrt{3}\sin(\omega t). \qquad\qquad (2)$$

(Check!! but don't hand in.)

We wish to model the temperature inside the house, assuming no sources or sinks for heat exist inside,

and using Newton's Law of Cooling. This model yields the linear DE

$$\frac{du}{dt} = k\,(A(t) - u)$$

for the indoor temperature u(t), i.e.

$$\frac{du}{dt} + k\,u = k\,(a_0 + a_1 \cos(\omega\,t) + b_1 \sin(\omega\,t)). \qquad (3)$$

The proportionality constant k reflects how quickly heat from the outside affects temperature change inside - the smaller k is, the more slowly the inside temperature changes for fixed difference $A(t) - u$, i.e. small k means good insulation.

Exercise 1. Using the integration factor algorithm we've learned for solving first order linear DEs, and working by hand, show that the general solution to the DE (1) above is given just as the book claims in equation 4, page 57. In other words, that the solutions are

$$u(t) = a_0 + c_0\,e^{(-k\,t)} + c_1 \cos(\omega\,t) + c_2 \sin(\omega\,t). \qquad (4)$$

Here the constant c_0 arises as a constant of integration, and is related to the initial temperature u_0. The constants c_1 and c_2 are determined by the insulation constant k and the angular frequency ω, as shown on page 57. The formulas you derive for c_0, c_1, c_2 agree with the book's display after equation (4). You may wish to use the book's integral tables, specifically #49,50 on the back cover of our text.

Exercise 2. Redo the computation for Exercise 1 using Maple to compute all the integrals you need. Your work here should all be done in the Maple worksheet: The Maple commands and output should be shown, along with any text explanation or command comments you find necessary.

Notice that the exponential part of the solution to (4) decays to zero (exponentially) as $t \to \infty$, and that what remains in the limit is also a solution (i.e. when the integration constant $c_0 = 0$). We call this the steady periodic (sp) solution, and write

$$u_{sp}(t) = a_0 + c_1 \cos(\omega\,t) + c_2 \sin(\omega\,t).$$

Exercise 3. Use Maple to crunch the numbers and show that with the Athens Georgia temperature data (i.e. the ambient temperature A(t) given by equation (2), and insulation constant

$$k = .2$$

we get

$$u(t) = 80 + (e^{(-.2\,t)})(u_0 - 82.3351) + 2.3351 \cos\left(\frac{1\,\pi\,t}{12}\right) - 5.6036 \sin\left(\frac{1\,\pi\,t}{12}\right) \qquad (5)$$

and so also,

$$u_{sp}(t) = 80 + 2.3351 \cos\left(\frac{1\,\pi\,t}{12}\right) - 5.6036 \sin\left(\frac{1\,\pi\,t}{12}\right) \qquad (6)$$

Here's a nice picture of the various solutions to the IVP for our DE, with the Athens data. In fact, it's figure 1.5.10 in the text. I've included the Maple commands which created the display. Notice the syntax elements of these commands - how to define functions, that := is how to define something in Maple, etc. Notice too that I had Maple check a lot of the work you'll be doing by hand above, using the "dsolve" command!!!!! (Although the answer is a mess it should agree with your hand work and the text after you do algebra.)

```
> restart: #clear all old definitions from memory
> with(DEtools):  #load the DE library of commands
> A:=t->a0 + a1*cos(omega*t) + b1*sin(omega*t);
    #formula for ambient temperature, with free
    #parameters a0, a1, b1, omega.  This is equation
    #(1), but not using subscripts on the variables.
    #When you enter multi-line
    #commands hold down the shift key while you hit
    #"enter" or "return", to prevent premature execution
```

$$A := t \rightarrow a0 + a1 \cos(\omega\, t) + b1 \sin(\omega\, t)$$

```
> deqtn3:=diff(u(t),t)=-k*(u(t)-A(t));
    #I named the DE "deqtn3" since it's equation (3)
    #on top of page 57.
```

$$deqtn3 := \frac{d}{dt} u(t) = -k\,(u(t) - a0 - a1 \cos(\omega\, t) - b1 \sin(\omega\, t))$$

```
> dsolve({deqtn3,u(0)=0},u(t));
```

$$u(t) = \frac{\mathbf{e}^{(-k\,t)}\,(-k^2\,a1 + k\,b1\,\omega - a0\,k^2 - a0\,\omega^2)}{k^2 + \omega^2}$$

$$+ \frac{a0\,k^2 + a0\,\omega^2 + k^2\,a1 \cos(\omega\, t) + k\,a1\,\omega \sin(\omega\, t) - k\,b1\,\omega \cos(\omega\, t) + k^2\,b1 \sin(\omega\, t)}{k^2 + \omega^2}$$

```
> a0:=80;
    #average ambient temp in Georgia in July
  a1:=-5;
  b1:=-5*sqrt(3.0);
    #the a1 and b1 values were worked out by hand,
    #using the cosine addition
    #angle formula, assuming 4 a.m. temp min and 4 p.m max,
    #and range from 70 to 90 degrees,
    #for trigonometric temp oscillation.
  omega:=Pi/12;
    #this makes the period equal to 24 (hours)
  k:=0.2;
    #constant for a well-insulated building
```
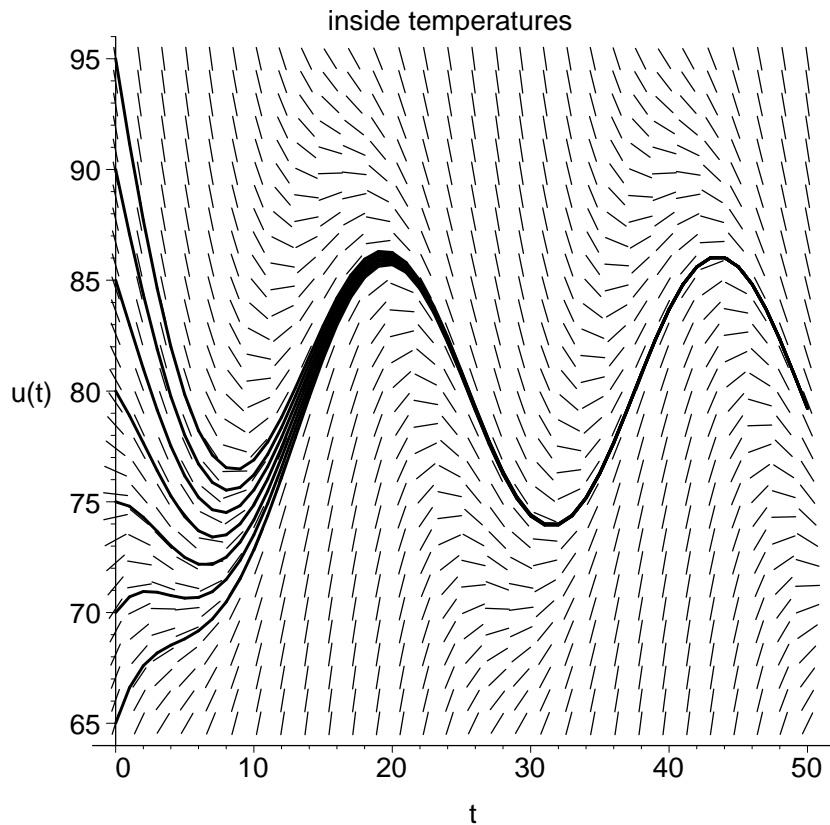
$$a0 := 80$$

$$a1 := -5$$

$$b1 := -8.660254040$$

$$\omega := \frac{\pi}{12}$$

$$k := 0.2$$

```
[ >
[ > DEplot(deqtn3,u(t),t=0..50,{[u(0)=65],[u(0)=70],
      [u(0)=75],[u(0)=80],[u(0)=85],[u(0)=90],[u(0)=95]},
       arrows=line, color=black,linecolor=black,
       dirgrid=[30,30], stepsize=1,
       title="inside temperatures");
```
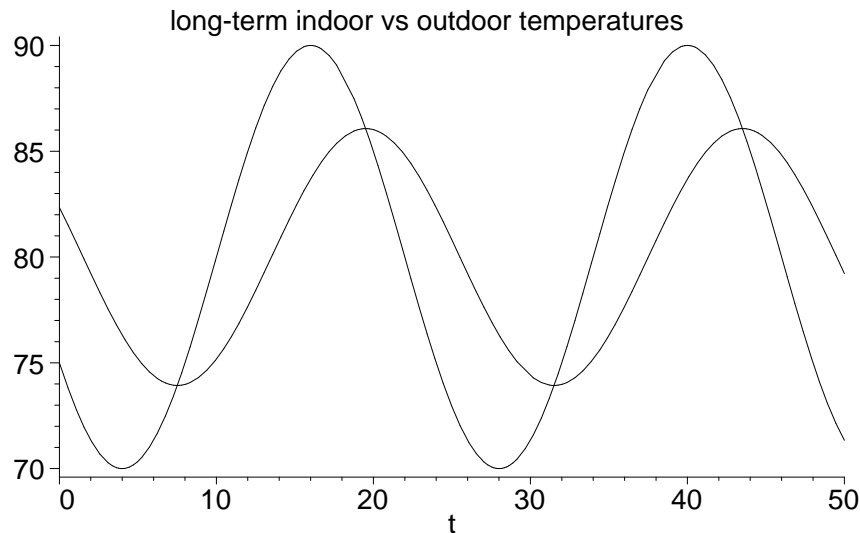

inside temperatures

We can also recreate a comparison of the steady-periodic indoor temperature vs. the outdoor temperatures, Figure 1.5.11 on page 58:

```
[ > with(plots):   #load the plotting package
[ > usp:=t->80+(2.3351)*cos(Pi/12*t)-(5.6036)*sin(Pi/12*t):
    plot1:=plot(usp(t),t=0..50,color=black):
    plot2:=plot(A(t),t=0..50,color=black):
    display({plot1,plot2},title='long-term indoor vs outdoor
    temperatures');
```

long-term indoor vs outdoor temperatures

```
[ >
```

Notice that the indoor temperatures have a smaller amplitude, and that they lag the outdoor temperatures. The same thing happens with seasonal temperatures on earth, where the temperatures in a given hemisphere lag behind the times of maximal and minimal heating from the sun.

Exercise 4: Use the cosine addition angle formula or Maple to show that the steady periodic solution given by equation (6) above, can be written in phase-amplitude form as

$$u_{sp}(t) = 80 - 6.0707 \cos\left(\frac{\pi(t - 7.5082)}{12}\right)$$

so that the coolest and hottest indoor temperatures are around 7:30 a.m. and p.m., with smaller amplitude of around 6.07, vs. the outdoor amplitude of 10 degrees.

Now we're in Salt Lake City, it's winter, we're worried about pipes freezing after our heater breaks, and we're doing exercises L2.3 and L2.4 from Professor Gustafson's project! So, the setup is that our heater is turned off at midnight, t=0. Perhaps we forgot to pay our natural gas bill. At this time, the indoor temperature is $u_0 = 74$ degrees. The outdoor ambient temperature is given as in equation (1), except that the 24-hour temperatures vary between 21 and 49 degrees, with the high and low at 3 p.m. and 3 a.m., respectively. We don't have a very well insulated house, its insulation constant is $k = .32$.

Exercise 5: From the information above deduce the constants in the formula for the ambient temperature A(t), and use Maple to find the solution to the initial value problem for the indoor temperature u(t).

Exercise 6: Create a Maple display which shows three plots: the Ambient temperature, the indoor temperature, and the steady-periodic indoor temperature.

Exercise 7: Find the phase delay (the difference, in hours), between the ambient temperature and the indoor steady periodic temperature. It is acceptable to approximate this from the display in Exercise 6 (clicking the mouse on a point of the display will yield the point coordinates in a small window at the top of your Maple program). It is even better if you can verify the phase delay by writing the

steady-periodic solution in phase-amplitude form (see section 5.4 mechanical vibrations, page 324, for help).

Exercise 8: Suppose the inside temperature is 76 degrees when the furnace is turned off. Using 3-d plots and other commands (see hints at Professor Gustafson's page, or use Maple help), figure out for the range of insulation constants $.2 < k < .48$, the times during the first 72 hours when the indoor temperature is at or below 30 degrees. (This could be bad for your water pipes.) Justify the logic and math used to find these "bad" times, in a short paragraph. Illustrate your work with a computer graphic.