Name..

Student I.D. number...

Math 2250-1

Practice Exam

November 14, 2008

The exam is closed book and closed note, but a scientific calculator is allowed and may be useful. Calculators which can do linear algebra or solve differential equations are not allowed. In order to receive partial credit or full credit all work must be shown. There are 100 points possible on the test, and the point values of each problem are indicated in the right margin. You may wish to ration your time accordingly. GOOD LUCK!

Solutions

1) Here is a matrix A:

$$\begin{array}{cc} & \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ A := & \begin{bmatrix} 1 & 3 & 2 & 5 & 5 \\ -1 & 1 & 2 & -1 & 3 \\ 2 & 0 & -2 & 4 & -2 \\ -2 & 2 & 4 & -2 & 5 \end{bmatrix} \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array} \end{array}$$

Here is its reduced row echelon form:

$$\mathrm{RREF}(A) := \begin{bmatrix} 1 & 0 & -1 & 2 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array}$$

1a) Find all solutions to the homogeneous linear equation Ax=0.

(10 points)

backsolve

$$\begin{aligned} x_1 &= s - 2t \\ x_2 &= -s - t \\ x_3 &= s \\ x_4 &= t \\ x_5 &= 0 \end{aligned}$$

$$\vec{x} = s \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

1b) Exhibit a basis for the solution space you found in part 1a), and explain why (prove) it's a basis. Your explanation should make it clear that you know what it means for a collection of vectors to be a basis!

(5 points)

a basis for this 2-dim'l space is $\left\{ \underset{\vec{v}_1}{\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}}, \underset{\vec{v}_2}{\begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}} \right\}$.

is basis because:

<u>span</u>: by construction, every solution to $A\vec{x} = \vec{0}$ is a linear combination of $\vec{v}_1$ & $\vec{v}_2$

<u>independent</u> If $c_1 \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Leftarrow$

then the 3rd component eqtn is $c_1 = 0$

& the 4th " " " $c_2 = 0$, so $c_1 = c_2 = 0$

$\blacksquare$

2) Consider the differential equation

$$y'''(x) - 2y''(x) + 5y'(x) = 0$$

2a) Find the general solution to this differential equation. (10 points)

for $y = e^{rx}$, $p(r) = r^3 - 2r^2 + 5r = 0$

$= r(\underbrace{r^2 - 2r + 5})$

$r = 0$

$y_1 = e^{0x} = 1$

$(r-1)^2 + 4 = 0$ or quad formula, $r = \frac{2 \pm \sqrt{4-20}}{2} = 1 \pm 4i$

$(r-1)^2 = -4$

$r - 1 = \pm 2i$

$r = 1 \pm 2i$

$e^{(1+2i)x} = e^x e^{2ix} = e^x(\cos 2x + i \sin 2x)$

$= \underset{y_2}{e^x \cos 2x} + i\, \underset{y_3}{e^x \sin 2x}$

$$\boxed{y_H = c_1 + c_2 e^x \cos 2x + c_3 e^x \sin 2x}$$

2b) Using your work from (2a) exhibit a basis for the solution space. Could you explain why your basis is a basis, if there had been room for more questions on this practice exam? (5 points)

basis $\{1, e^x \cos 2x, e^x \sin 2x\}$

I could explain!

ans 1) the soltn space is 3-dim'l because the DE is 3rd order linear homogeneous (theorem). If $\{1, e^x \cos 2x, e^x \sin 2x\}$ are ind. they automatically span. Check ind with Wronskian, (or other ways)

$$W = \det \begin{bmatrix} 1 & e^x \cos 2x & e^x \sin 2x \\ 0 & e^x(-2\sin 2x + \cos 2x) & e^x(2\cos 2x + \sin 2x) \\ 0 & e^x(-4\sin 2x - 3\cos 2x) & e^x(4\cos 2x - 3\sin 2x) \end{bmatrix}$$

$@\, x = 0$ $W = \det \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & -3 & 4 \end{bmatrix} = 1 \begin{vmatrix} 1 & 2 \\ -3 & 4 \end{vmatrix} = 10 \neq 0$

ans 2) since the Wronskian matrix

$[W] = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & -3 & 4 \end{bmatrix}$ is invertible, I can solve the IVP

for $L(y) = 0$, $y(0) = b_1$, $y'(0) = b_2$, $y''(0) = b_3$ with $y = c_1 y_1 + c_2 y_2 + c_3 y_3$

where $[W(0)]\vec{c} = \vec{b}$

$\vec{c} = [W(0)]^{-1}\vec{b}$

this shows span! directly

2c) Find the general solution to the nonhomogeneous linear differential equation

$$y'''(x) - 2y''(x) + 5y'(x) = 50x + 20e^{(2x)}$$

(10 points)

need y_p.

for $Ly_p = 50x$

try

$+0(\ y_p = x(Ax+B)\ )$ because B solves homog eqtn.

$+5(\ y_p' = 2Ax+B\ )$

$-2(\ y_p'' = 2A\ )$

$1(\ y_p''' = 0\ )$

$Ly_p = 10Ax + 5B - 4A = 50x$

$\left.\begin{array}{l} 10A = 50 \\ 5B - 4A = 0 \end{array}\right\}$ so $\begin{array}{l} A = 5 \\ B = 4 \end{array}$

$y_p = 5x^2 + 4x$

for $Ly_p = 20e^{2x}$

try $y_p = Ae^{2x}$

$+5(\ y_p' = 2Ae^{2x}\ )$

$-2(\ y_p'' = 4Ae^{2x}\ )$

$1(\ y_p''' = 8Ae^{2x}\ )$

$Ly_p = A[10 - 8 + 8]e^{2x} = 20e^{2x}$

$A = 2$

$y_p = 2e^{2x}$

So by superposition,

$$\boxed{y = 5x^2 + 4x + 2e^{2x} + c_1 + c_2 e^x \cos 2x + c_3 e^x \sin 2x}$$

3) Consider this differential equation, which could arise as a mass-spring (or electrical circuit) problem.

$$x''(t) + c\,x'(t) + 3\,x(t) = 0$$

3a) Identify possible values for mass, damping coefficient and Hooke's (spring) constant which could yield the differential equation above. Make sure to include correct units.

(6 points)

$m = 1$ kg
$c = c$ N s/m (kg/s)
3 N/m

3b) Give the precise values or intervals of c which correspond to undamped, underdamped, critically damped, and overdamped problems, for the differential equation above.

(4 points)

$x = e^{rt} \rightarrow p(r) = r^2 + cr + 3 = 0$

$r = \dfrac{-c \pm \sqrt{c^2 - 12}}{2}$

$c > \sqrt{12} \rightarrow$ real roots overdamped
$c = \sqrt{12} \rightarrow$ double real root critical damped
$c < \sqrt{12} \rightarrow$ complex roots
overdamped

3c) Now consider the forced oscillation problem,

$$x''(t) + c\,x'(t) + 3\,x(t) = F_0 \cos(\omega t)$$

What values of "c" and "omega" would lead to resonance? What form would you guess for a particular solution, in this case? DO NOT ACTUALLY FIND THE PARTICULAR SOLUTION (because I have no points to spare on this practice exam).

(5 points)

$c = 0$ so $x'' + 3x = F_0 \cos \omega t$
$\omega = \omega_0 = \sqrt{3}$

then try $\boxed{\vec{x}_p(t) = t\,(A \cos \omega_0 t + B \sin \omega_0 t)}$
(turns out $A = 0$).

3d) Now consider the damped forced oscillator

$$x''(t) + 4x'(t) + 3x(t) = -30\sin(3t)$$

This differential equation has a particular solution

$$x_p(t) = 2\cos(3t) + \sin(3t)$$

Use this particular solution along with the solution to the homogeneous problem (which you must find), in order to solve the initial value problem

$$x''(t) + 4x'(t) + 3x(t) = -30\sin(3t)$$
$$x(0) = 2$$
$$x'(0) = -1$$

Identify the steady periodic and transient pieces of the solution.

(15 points)

given $x_p(t)$!

so only need $x_H(t)$: $r^2 + 4r + 3 = 0$

$(r+3)(r+1) = 0$

$$x_H(t) = c_1 e^{-3t} + c_2 e^{-t}$$

$$x(t) = 2\cos 3t + \sin 3t + c_1 e^{-3t} + c_2 e^{-t}$$
$$x'(t) = -6\sin 3t + 3\cos 3t - 3c_1 e^{-3t} - c_2 e^{-t}$$

$x(0) = 2 = 2 + c_1 + c_2$

$x'(0) = -1 = 3 - 3c_1 - c_2$

$c_1 + c_2 = 0$

$-3c_1 - c_2 = -4$

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ -3 & -1 & -4 \end{array}\right]$$

$3R_1 + R_2$
$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 2 & -4 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & -2 \end{array}\right]$$

$-R_2 + R_1$
$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -2 \end{array}\right]$$

$c_1 = 2$

$c_2 = -2$

$$x(t) = \underbrace{2\cos 3t + \sin 3t}_{x_{sp}(t)} + \underbrace{2e^{-3t} - 2e^{-t}}_{x_{tr}(t)}$$

4) Resolve the initial value problem in (3d) using Laplace transform. For your convenience, I repeat it here:

$$x''(t) + 4x'(t) + 3x(t) = -30\sin(3t)$$
$$x(0) = 2$$
$$x'(0) = -1$$

(20 points)

$$s^2 X(s) - s\cdot \underbrace{2}_{x(0)} - \underbrace{-1}_{x'(0)} + 4(sX(s) - \underbrace{2}_{x(0)}) + 3X(s) = -30\cdot\frac{3}{s^2+9}$$

$$X(s)[s^2+4s+3] = \frac{-90}{s^2+9} + 2s + 7$$

$$X(s) = \frac{-90}{(s^2+9)(s^2+4s+3)} + \frac{2s+7}{s^2+4s+3}$$

$$= \frac{-90}{(s^2+9)(s+3)(s+1)} + \frac{2s+7}{(s+3)(s+1)}$$

$$= \frac{As+B}{s^2+9} + \frac{C}{s+3} + \frac{D}{s+1}$$

$$-90 = (As+B)(s+3)(s+1) + C(s^2+9)(s+1) + D(s^2+9)(s+3)$$

$s=-1$: $-90 = D(10)(2)$ $\boxed{D = -\frac{9}{2}}$

$s=-3$: $-90 = C(18)(-2)$

$= C(9)(-4)$

$-10 = -4C$ $\boxed{C = \frac{5}{2}}$

coeff of s^3: $0 = A + C + D$

$= A + \frac{5}{2} - \frac{9}{2}$

$\boxed{A = +2}$ (agrees with previous page!)

coeff of 1: $3B + 9C + 27D = -90$

$3B + \frac{45}{2} - 27\left(\frac{9}{2}\right) = -90$

$3B + 9\left(\frac{5}{2} - \frac{27}{2}\right) = -90$

$3B - 9\cdot 11 = -90$

$3B = 9$

$\boxed{B=3}$

whew!

$$= \frac{E}{s+3} + \frac{F}{s+1}$$

$$2s+7 = E(s+1) + F(s+3)$$

$s=-1$ $5 = 2F$ $\boxed{F = \frac{5}{2}}$

$s=-3$ $1 = -2E$ $\boxed{E = -\frac{1}{2}}$

All together,

$$X(s) = \frac{2s}{s^2+9} + \frac{3}{s^2+9} + \frac{E+C}{s+3} + \frac{D+F}{s+1}$$

$$= \frac{2s}{s^2+9} + \frac{3}{s^2+9} + \frac{2}{s+3} + \frac{-2}{s+1}$$

so

$$x(t) = 2\cos 3t + \sin 3t + 2e^{-3t} - 2e^{-t}$$

Yipes!!

5a) Find the function x(t) which has Laplace transform

$$X(s) = \frac{2s+1}{s^2+4s+20}.$$

(5 points)

$$= \frac{2(s+2)-3}{(s+2)^2+16}$$

← then complete the linear

← complete the square first

$$= 2\underbrace{\frac{s+2}{(s+2)^2+16}}_{= \mathcal{L}\{\cos 4t\}(s+2)} - \frac{3}{4}\underbrace{\frac{4}{(s+2)^2+16}}_{= \mathcal{L}\{\sin 4t\}(s+2)}$$

so by translation theorem

$$\boxed{x(t) = 2e^{-2t}\cos 4t - \frac{3}{4}e^{-2t}\sin 4t}$$

5b) Use the convolution definition to compute (f*g)(t), for f(t)=t and g(t)= t^2. Then use the Laplace transform table to check that your answer has Laplace transform which is the product of the Laplace transforms of f(t) and g(t).

(5 points)

$$(f*g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau = \int_0^t g(\tau)f(t-\tau)\,d\tau$$

↓ easier

$$\int_0^t \tau^2(t-\tau)\,d\tau$$

$$= t\underbrace{\int_0^t \tau^2\,d\tau}_{\left.\frac{\tau^3}{3}\right]_0^t} - \underbrace{\int_0^t \tau^3\,d\tau}_{\left.\frac{\tau^4}{4}\right]_0^t}$$

$$= \frac{t^4}{3} - \frac{t^4}{4} = \boxed{\frac{t^4}{12}}$$

check $\mathcal{L}\{t\}(s) = \frac{1}{s^2}$

$\mathcal{L}\{t^2\}(s) = \frac{2}{s^3}$

$$\mathcal{L}\left\{\frac{t^4}{12}\right\}(s) = \frac{4!}{12 \cdot s^5} = \frac{24}{12 s^5} = \frac{2}{s^5}$$

$$\frac{1}{s^2} \cdot \frac{2}{s^3} = \frac{2}{s^5}$$

$F(s)\,G(s) = \mathcal{L}\{f*g\}(t)(s)$ ✓