Name...

I.D. number...

Math 2250-1
Practice Exam 1 -Solutions
October 2008

This exam is closed-book and closed-note. You may use a scientific calculator, but not one which is capable of graphing or of solving differential or linear algebra equations. **In order to receive full or partial credit on any problem, you must show all of your work** and **justify your conclusions.** There are 100 points possible. The point values for each problem are indicated in the right-hand margin.
Good Luck!

1) Consider a brine tank which holds 4000 gallons of continuously-mixed liquid. Let x(t) be the amount of salt (in pounds) in the tank at time t. The in-flow and out-flow rates are both 200 gallons/hour, and the concentration of salt flowing in is 0.02 pounds per gallon.
1a) Use the information above and your modeling ability to derive the differential equation for x(t):

$$\frac{dx}{dt} = 4 - \frac{1\,x}{20}$$

(5 points)

The input-output model is

$$\frac{dx}{dt} = r_i\, c_i - r_o\, c_0$$

In this problem, the incoming rate is $r_i = 200$ *(gallons/minute), and so it the outgoing rate* r_o *. We are also told that the incoming concentration is* $c_i = 0.02$ *(pounds/gallon). Under the well-mixed assumption the outgoing concentration is* $c_o = \frac{x}{4000}$ *. Thus,*

$$\frac{dx}{dt} = [200]\,[0.02] - [200]\left[\frac{x}{4000}\right]$$

$$= 4 - \frac{x}{20}\ .$$

1b) What is the equilibrium solution to this differential equation? Is it stable or unstable? Explain.

(5 points)

The equilibrium solution is the constant solution, i.e. the one with time derivative equal to zero. Thus the right side of the DE must also be zero, so

$$4 - \frac{x}{20} = 0$$

$$x = 80 \text{ gal/min}$$

Since $x'(t) = 4 - \frac{x}{20} = \left[\frac{1}{20}\right][80 - x]$ *, we see that x(t) is decreasing for x>80, and increasing for x<80. Thus x=80 is a stable equilibrium (I'd draw a phase portrait but don't know how to do that in Maple.)*

1c) Find the solution to the initial value problem, assuming their are originally 20 pounds of salt in the tank.

(15 points)

This problem is both linear and separable, so you should be able to solve it either way....Here, for example, is the "linear" algorithm:

$$\frac{dx}{dt} + 0.05\, x = 4$$

$$\mathrm{D}_t\,[\mathbf{e}^{(0.05\,t)}\,\mathrm{x}(t)] = 4\,\mathbf{e}^{(0.05\,t)}$$

$$\mathbf{e}^{(0.05\,t)}\,\mathrm{x}(t) = \int 4\,\mathbf{e}^{(0.05\,t)}\,dt$$

$$\mathrm{x}(t) = 80 + C\,\mathbf{e}^{(-0.05\,t)}.$$

Since x(0)=20, C must equal -60;

$$\mathrm{x}(t) = 80 - 60\,\mathbf{e}^{(-0.05\,t)}.$$

1d) Is the limiting value of x(t) as t approaches infinity consistent with your discussion in (1b)? How could your smart little sister have told you the limting amount of salt without understanding any differential equations, i.e. in terms of the in-flow concentration?

(5 points)

The limit of x(t) as t approaches infinity is 80 pounds. She might has said that the eventual concentration would approximate the incoming concentration, which was .02 pounds per gallon. Since the tank holds 4000 gallons, this would yield (.02)(4000) = 80 pounds of salt in the limit.

2) A certain population P(t) of animals satisfies a logistic growth differential equation, and this population is harvested at a constant rate, leading to the differential equation

$$\frac{dP}{dt} = -P^2 + 4\,P - 3$$

2a) Assume that the units of P(t) are thousands of animals, and that time is measured in years. At what rate are the animals being harvested?

(4 points)

The number "3" is the constant harvesting rate (and the $-P^2 + 4\,P$ *is the logistic harvest rate). Thus the animals are being harvested at 3000 animals per year.*

2b) What are the equilibrium populations for this differential equation? Make a phase portrait and determine which equilibria are stable and which are unstable.

(8 points)

Equilibrium solutions are constant solutions, i.e. $\frac{dP}{dt}$ *is zero, so after writing*

$$-P^2 + 4\,P - 3 = -(P^2 - 4\,P + 3) = -(P-3)(P-1),$$

we see that $P = 1$ *and* $P = 3$ *are the equilibrium solutions. I don't know how to make a phase portrait with Maple, but using the factoring of the right side above, we see that P is decreasing for P>3 and increasing for 1<P<3. P is decreasing for P<1. Thus P=3 is a stable equilibrium and P=1 is undstable.*

2c) If the initial population was 2000 animals ($P_0 = 2$), what do you expect the limiting population to be, based upon your work in (2b)?

(4 points)

Since P is increasing for 1<P<3, we expect the solution to the IVP with $P_0 = 2$ *to converge to P=3 (thousand) as t approaches infinity.*

2d) Solve the initial value problem for this harvested logistic equation, with $P_0 = 2$.

(15 points)

This DE is separable:

$$\frac{dP}{(P-3)(P-1)} = -dt$$

$$\left(\frac{1}{P-3} - \frac{1}{P-1}\right) dP = -2\,dt$$

$$\ln\left(\left|\frac{P-3}{P-1}\right|\right) = -2\,t + C_1$$

$$\left|\frac{P-3}{P-1}\right| = \mathbf{e}^{C_1}\,\mathbf{e}^{(-2\,t)}$$

$$\frac{P-3}{P-1} = C\,\mathbf{e}^{(-2\,t)}$$

Since P(0)=2, we see that $-1 = C$.

$$\frac{P-3}{P-1} = -\mathbf{e}^{(-2\,t)}$$

$$\mathrm{P}(1 + \mathbf{e}^{(-2\,t)}) = 3 + \mathbf{e}^{(-2\,t)}$$

$$P - 3 = -\mathbf{e}^{(-2\,t)}\,(P-1)$$

$$P = \frac{3 + \mathbf{e}^{(-2\,t)}}{1 + \mathbf{e}^{(-2\,t)}} .$$

2e) Is your predicted limiting population from (2b) consistent with your actual solution from (2d)?

(4 points)

Yes, the limit is 3, since the exponential terms limit to zero.

3) Consider the matrix A defined by

$$A := \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & 1 \\ -2 & 3 & -6 \end{bmatrix}$$

3a) Compute the determinant of A.

(4 points)

If I cross-hatch I get -6+2+0--6-3-0=-1. If I expand down the first column I get 1(-6-3)-0+(-2)(-1-4)=-9+8=-1. You should get the same answer no matter which technique you use!

3b) What does the value of the determinant you computed in (4a) tell you about whether or not the matrix A has an inverse? Explain, by quoting the general fact relating determinant values to matrix inverses.

(4 points)

Since the determinant is non-zero the inverse matrix exists.

3c) Find the inverse matrix to A, using the row-operation algorithm. Remember to show all work, as always. Hint: The correct inverse matrix has no fractions in it, which you could deduce from your work in the answer to part (3a)

(12 points)

The algorithm is to augment A with the identity matrix, compute rref, and see the inverse matrix to the right of the identity matrix. Precisely

$$AaugId := \begin{bmatrix} 1 & -1 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ -2 & 3 & -6 & 0 & 0 & 1 \end{bmatrix}$$

which (after several steps) reduces to

$$\begin{bmatrix} 1 & 0 & 0 & 9 & -3 & 4 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & -2 & 1 & -1 \end{bmatrix}$$

So the inverse matrix is

$$B := \begin{bmatrix} 9 & -3 & 4 \\ 2 & 0 & 1 \\ -2 & 1 & -1 \end{bmatrix}$$

3d) Recompute the inverse, using the adjoint formula. Show the cofactor matrix as well as the adjoint matrix as intermediate steps, so that I can check your work.

(10 points)

$$cof := \begin{bmatrix} \mathrm{Det}\left(\begin{bmatrix} 1 & 1 \\ 3 & -6 \end{bmatrix}\right) & -\mathrm{Det}\left(\begin{bmatrix} 0 & 1 \\ -2 & -6 \end{bmatrix}\right) & \mathrm{Det}\left(\begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}\right) \\ -\mathrm{Det}\left(\begin{bmatrix} -1 & 3 \\ 3 & -6 \end{bmatrix}\right) & \mathrm{Det}\left(\begin{bmatrix} 1 & 3 \\ -2 & -6 \end{bmatrix}\right) & -\mathrm{Det}\left(\begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}\right) \\ \mathrm{Det}\left(\begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix}\right) & -\mathrm{Det}\left(\begin{bmatrix} 1 & 3 \\ -2 & -6 \end{bmatrix}\right) & \mathrm{Det}\left(\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\right) \end{bmatrix}$$

$$= \begin{bmatrix} -9 & -2 & 2 \\ 3 & 0 & -1 \\ -4 & -1 & 1 \end{bmatrix}.$$

So the adjoint matrix, which is the transpose of the cofactor matrix is given by

$$adj = \begin{bmatrix} -9 & 3 & -4 \\ -2 & 0 & -1 \\ 2 & -1 & 1 \end{bmatrix}.$$

So, since the inverse is 1/det(A) times the adjoint, and since det(A)=-1, we get that the inverse of A is

$$\begin{bmatrix} 9 & -3 & 4 \\ 2 & 0 & 1 \\ -2 & 1 & -1 \end{bmatrix}.$$

3e) Use the inverse matrix from part (3c) or (3d) to solve the system

$$\begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & 1 \\ -2 & 3 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ -11 \end{bmatrix}$$

(5 points)

The solution is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 & -3 & 4 \\ 2 & 0 & 1 \\ -2 & 1 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ 0 \\ -11 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$