Math 2250
Tuesday Oct 7.

Recall our discussion from Friday, and the language we are using:

If $\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n\}$ are vectors then

a <u>linear combination</u> of $\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n$ is $\boxed{}$

the <u>linear combination coefficients</u> are $\boxed{}$

the span of $\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n\}$ is $\boxed{}$

On Friday we did examples in $\mathbb{R}^3$ where the span of different sets of vectors turned out to be a line, plane, or all of $\mathbb{R}^3$.

Sometimes the linear combination coeff's were unique, and sometimes they weren't - this turns out to be important, and is related to the concept of <u>dependence</u> / <u>independence</u>:

<u>Definition</u>

a) the set $\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n\}$ is <u>linearly dependent</u> (or the vectors $\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n$ are <u>linearly dependent</u>)

iff some linear combination
$$c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n = \vec{0}$$
where not all the coeficients $c_j = 0$.

(<u>Note</u>: this is a precise way of saying that at least one of the $\vec{v}_j$'s is a linear combination of the other $\vec{v}_k$'s.)

b) $\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n$ are <u>linearly independent</u> iff the only linear combination
$$c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n = \vec{0}$$
is the one where each $c_j = 0$.

(this is a precise way of saying that no $\vec{v}_j$ is a linear combination of the others.)

<u>Example</u> 1 (from Friday): $\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -1 \\ 1 \\ -5 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 4 \\ -1 \\ 11 \end{bmatrix}$

are not linearly independent (they are linearly dependent), because
$$\vec{v}_3 = \vec{v}_1 - 2\vec{v}_2 \qquad (\text{i.e. } \vec{v}_1 - 2\vec{v}_2 - \vec{v}_3 = \vec{0})$$

Example 2 Are $\vec{e}_1, \vec{e}_2, \vec{e}_3$ linearly dependent or independent?

$$\vec{e}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, \quad \vec{e}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}, \quad \vec{e}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

As we saw on Friday, for example 2, the linear combo coefficients are uniquely determined for any $\vec{v}$ in the span $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ $(= \mathbb{R}^3)$
and for example 1, they are not uniquely determined for $\vec{w} \in \text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$
(= the plane $-2x+3y+z=0$)

These are examples of:

Theorem

The linear combination coefficients of each $\vec{w} \in \text{span}\{\vec{v}_1, \ldots \vec{v}_k\}$ are unique if and only if $\{\vec{v}_1, \ldots \vec{v}_k\}$ is linearly independent.

proof: If
$$\vec{w} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_k\vec{v}_k$$
$$\vec{w} = b_1\vec{v}_1 + b_2\vec{v}_2 + \cdots + b_k\vec{v}_k$$

then
$$\vec{0} = \vec{w} - \vec{w} = (a_1-b_1)\vec{v}_1 + (a_2-b_2)\vec{v}_2 + \cdots + (a_k-b_k)\vec{v}_k$$
$$= c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k \qquad (c_j = a_j - b_j)$$

So if $\{\vec{v}_1, \ldots \vec{v}_k\}$ is linearly independent each $c_j = 0$, so each $a_j = b_j$ and the linear combo coefficients of $\vec{w}$ are unique.
Conversely, if linear combo coefficients are unique, then the only way to express $\vec{0}$ is $\vec{0} = 0\vec{v}_1 + 0\vec{v}_2 + \cdots + 0\vec{v}_k$.
So $\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_k\}$ are linearly independent ∎

examples from HW

§ 4.3 #1 : Are $\vec{v}_1 = \begin{bmatrix} 4 \\ -2 \\ 6 \\ -4 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 6 \\ -3 \\ 9 \\ -6 \end{bmatrix}$ linearly independent or dependent?

What geometric object is span$\{\vec{v}_1, \vec{v}_2\}$?

§ 4.3 #3 : Are $\vec{v}_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 6 \\ -1 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$ linearly independent or dependent?

What geometric object is span$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$?

use rref to answer these questions: Explain!!

Can more than n vectors in $\mathbb{R}^n$ be linearly independant?
Why not?

Can less than n vectors in $\mathbb{R}^n$ span all of $\mathbb{R}^n$?
Why not?

If you have exactly n vectors in $\mathbb{R}^n$ what tests determine
whether they are linearly independent?
whether they span $\mathbb{R}^n$?

There are objects other than vectors in $\mathbb{R}^n$ which one can add and scalar multiply, and for which the expected arithmetic rules apply. Thus we will be able to consider concepts like "span" and linear independence / dependence in these other settings as well.

The main example to consider here is

$$\mathcal{F} = \text{the set of real-valued functions with domain } \mathbb{R} = \{ f : \mathbb{R} \to \mathbb{R} \}.$$

where function addition and scalar multiplication are defined as in Calculus:

$$(f+g)(x) := f(x) + g(x)$$
$$(cf)(x) : c \cdot f(x)$$

⟵ functions are "vectors"

In $\mathbb{R}^n$ and in $\mathcal{F}$, addition and scalar mult satisfy the <u>vector space axioms</u>:

As set V of "vectors", together with operations $+$, scalar multiplication is called a <u>vector space</u> if the following axioms hold

(α) whenever $\vec{u}, \vec{v} \in V$ then $\vec{u} + \vec{v} \in V$ (closure wrt addition)

(β) whenever $\vec{u} \in V$, $k \in \mathbb{R}$, then $k\vec{u} \in V$ (" " scalar multiplication)

(a) $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ commutative

(b) $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$ associative

(c) $\vec{u} + \vec{0} = \vec{0} + \vec{u} = \vec{u}$ zero vector exists in V

(d) $\vec{u} + (-\vec{u}) = \vec{0}$ additive inverses exist

(e) $a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$ distributive prop

(f) $(a+b)\vec{u} = a\vec{u} + b\vec{u}$ "

(g) $a(b\vec{u}) = (ab)\vec{u}$

(h) $1\vec{u} = \vec{u}$

- What is the zero "vector" in $\mathcal{F}$?

- Are the "vectors" $\{1, x, x^2\}$ linearly independent in $\mathcal{F}$? What is their span?

Examples of vector spaces

1. $V = \mathbb{R}^n$ as we've been doing
2. $\mathcal{F}$ = real valued fcns with domain $\mathbb{R}$
   - we will use this vector space a lot when we return to differential eqtns
3. Subspaces W of a vector space V

   $\downarrow$

   a subset of V that is a vector space itself, via the V operations.

   to check whether W is a subspace, you need only check
   (α) closure wrt $+$
   (β) closure wrt scalar mult.

   Then (a) – (h) are basically inherited from V.

Important subspaces

- The solution set W to a homogeneous matrix equation $A\vec{x} = \vec{0}$
  i.e. $\{ \vec{x} \in \mathbb{R}^n \text{ s.t. } A\vec{x} = \vec{0} \}$ for a given $A_{m \times n}$

  Check this is a subspace of $\mathbb{R}^n$:

  (α) If $\vec{x}, \vec{y} \in W$ then $A\vec{x} = \vec{0}$, $A\vec{y} = \vec{0}$
  so $A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y} = \vec{0}$. Hence $\vec{x} + \vec{y} \in W$.

  (β) If $\vec{x} \in W$ then $A\vec{x} = \vec{0}$
  so $A(k\vec{x}) = kA\vec{x} = \vec{0}$ too.
  so $k\vec{x} \in W$.

- $W = \text{span}\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_k\}$.

  (α) If $\vec{x} = c_1\vec{v}_1 + \dots + c_k\vec{v}_k$, $\vec{y} = d_1\vec{v}_1 + \dots + d_k\vec{v}_k$ Then $\vec{x} + \vec{y}$
  $= (c_1 + d_1)\vec{v}_1 + \dots + (c_k + d_k)\vec{v}_k$
  $\in W$.

  (β) If $\vec{x} = c_1\vec{v}_1 + \dots + c_k\vec{v}_k$ then $k\vec{x} = (c_1 k)\vec{v}_1 + \dots + (c_k k)\vec{v}_k \in W$

Try some hw from
§4.2, e.g. 6, 9, 15.