So much for unforced (homogeneous) spring problems. Now discuss general non-homogeneous linear DE, then in §5.6 we'll return to springs:

yesterday → ~~5~~

today → 1

Math 2250-1
Wednesday Oct 29

§5.5 : Finding y_p's for $\mathcal{L}(y)=f$.

Recall, if $\mathcal{L}$ is a linear operator the general sol'n to

$$\mathcal{L}(y)=f$$

is $\quad y = y_P + y_H$

where y_p is any particular sol'n and
y_H is the general (n-dim'l) solution to $\mathcal{L}(y)=0$

How to find y_p's.

Book calls the method "undetermined coefficients",
for constant coefficient linear DE's,
is really just "guessing" (but $\underbrace{\text{could be}}_{\text{is!}}$ justified with vector space theory).

example 1

$$\mathcal{L}(y) = y'' + 3y' + 4y$$

solve $\quad \mathcal{L}(y) = 3x+2$

try $\quad y_p = Ax+B \qquad$ (because when you apply $\mathcal{L}$ to polys of degree ≤ 1 you get back polys of degree ≤ 1).

$$\begin{aligned} 4\,(\quad & y_p = Ax+B \\ +\,3\,(\quad & y_p' = A \\ +\,1\,(\quad & y_p'' = 0 \end{aligned}$$

$$\mathcal{L}(y_p) = x(4A) + 1(3A+4B) = 3x+2$$

$$\left.\begin{aligned} 4A&=3 \\ 3A+4B&=2 \end{aligned}\right\} \qquad \begin{aligned} A&=3/4 \\ \tfrac{9}{4}+4B&=2 \\ 4B&=-\tfrac{1}{4} \\ B&=-\tfrac{1}{16} \end{aligned}$$

$$\boxed{y_p = \tfrac{3}{4}x - \tfrac{1}{16}}$$

y_H: $\quad r^2+3r+4=0$

$$r = \frac{-3 \pm \sqrt{9-16}}{2} = -\frac{3}{2} \pm i\frac{\sqrt{7}}{2}$$

$$\boxed{y_H(x) = e^{-3/2\,x}\left(A\cos\tfrac{\sqrt{7}}{2}x + B\sin\tfrac{\sqrt{7}}{2}x\right)}$$

$$\boxed{y(x) = \tfrac{3}{4}x - \tfrac{1}{16} + e^{-3/2\,x}\left(A\cos\tfrac{\sqrt{7}}{2}x + B\sin\tfrac{\sqrt{7}}{2}x\right)}$$

Tomorrow, we'll continue the "method of guessing" for right-hand sides of certain types. See table p.341
(poly, exps, trig, exp-trig)

example 2 $y'' - 4y = 2e^{3x}$

find y_p:

try $y_p = Ae^{3x}$

$L(e^{3x}) = 9e^{3x} - 4e^{3x} = 5e^{3x}$

so $L(Ae^{3x}) = 5Ae^{3x}$

$\qquad\qquad\qquad \| $

$\qquad\qquad\qquad 2$; $A = \frac{2}{5}$ $\qquad y_p = \frac{2}{5}e^{3x}$

example 3 $y'' - 4y = 10e^{2x}$

find y_p

try

oh oh. → the RHS was related to y_H. There's a way to fix your guess:

example 4 find y_p to $y'' - 4y = 4e^{3x} + 5e^{2x}$

ans: (hint: look at the 2 examples above) guessing rules

If $L(y)$ is constant coeff. linear differential operator, what would be your first guess at the form of y_p if you wanted to solve the following, and why?

$L(y) = 3\cos 2x$

$y_p =$

$L(y) = 4e^{2x}\sin 3x$

$y_p =$

$L(y) = x^3 + 6x^2 - 5$

$y_p =$

$L(y) = 7e^{13x}$

$y_p =$

$L(y) = x\sin 2x$

$y_p =$

Remark: The linear algebra reason that "guessing" works: If V is finite-dim'l and $L: V \to V$ has nullspace $= \{0\}$ (only soln to $Lv = 0$ is $v = 0$), then $Lv = f$ has a unique soln v_p.

there are special rules if $f(x) = \text{RHS}$ is related to y_H: you multiply your guess for y_p by x^s, where s is the lowest natural power making each term in your guess NOT a solution of the homog. eqtn

See table page 341

If guessing won't work (maybe your linear DE is <u>not</u> const coeff) there is a method, <u>variation of parameters</u>, that will.

illustrate method on 2^{nd} example page 2:

$$y'' - 4y = 10e^{2x}$$

[by improved "guessing" we got

$$y_p = \tfrac{5}{2} x e^{2x}$$
$$y = \tfrac{5}{2} x e^{2x} + c_1 e^{2x} + c_2 e^{-2x}].$$

<u>Variation of parameters</u> : to find y_p if you have a basis $\{y_1, y_2\}$ for y_H

$$\mathcal{L}(y) = y'' + p(x) y' + q(x) y$$

$$y_H = \text{span}\{y_1, y_2\} \quad \text{known.}$$

to solve

$$\mathcal{L}(y) = f$$

try $y_p = u_1 y_1 + u_2 y_2$ $\qquad u_1, u_2 =$ fcns of x

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f \end{bmatrix}$$

$$\begin{aligned}
q(x)\,(\quad & y_p = u_1 y_1 + u_2 y_2 \\
p(x)\,(\quad & y_p' = u_1 y_1' + u_2 y_2' + \underbrace{u_1' y_1 + u_2' y_2}_{\text{set } = 0} \\
1\,(\quad \Rightarrow\ & y_p'' = u_1 y_1'' + u_2 y_2'' + \underbrace{u_1' y_1' + u_2' y_2'}_{\text{set } = f}
\end{aligned}$$

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \frac{1}{W} \begin{bmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{bmatrix} \begin{bmatrix} 0 \\ f \end{bmatrix}$$

so $\mathcal{L}(y_p) = u_1 \underbrace{\cancel{\mathcal{L}(y_1)}}_{0} + u_2 \underbrace{\cancel{\mathcal{L}(y_2)}}_{0.} + f$

$= f$

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \frac{1}{W} \begin{bmatrix} -y_2 f \\ y_1 f \end{bmatrix}$$

integrate to get u_1 & u_2 then get y_p.

$y_1 = e^{2x}$

$y_2 = e^{-2x}$

$$W = \begin{vmatrix} e^{2x} & e^{-2x} \\ 2e^{2x} & -2e^{-2x} \end{vmatrix} = -4$$

$f = 10e^{2x}$

$$u_1' = -\tfrac{1}{4}\left(-e^{-2x}\, 10 e^{2x}\right) = \tfrac{5}{2}$$
$$u_2' = -\tfrac{1}{4}\left(e^{2x}\, 10 e^{2x}\right) = -\tfrac{5}{2} e^{4x}$$

take $u_1 = \tfrac{5}{2} x$

$u_2 = -\tfrac{5}{8} e^{4x}$

$$y_p = u_1 y_1 + u_2 y_2$$
$$= \tfrac{5}{2} x e^{2x} - \tfrac{5}{8} e^{4x} e^{-2x}$$

$\boxed{y_p = \tfrac{5}{2} x e^{2x} - \tfrac{5}{8} e^{2x}}$, (agrees since $-\tfrac{5}{8} e^{2x}$ is a y_H)