Math 2250-1
Tuesday Oct 28

§5.4 cont'd.

We are studying unforced mechanical vibrations

$$m x'' + c x' + k x = 0$$

On Monday we studied

<u>Case 1</u> Free undamped motion ($c=0$)

$$m x'' + k x = 0$$

$$x'' + \omega_0^2 x = 0 \qquad \omega_0 := \sqrt{k/m}$$

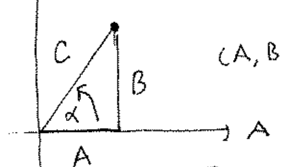
(review; we won't go over this in class)

sol'n

$$x(t) = A \cos \omega_0 t + B \sin \omega_0 t$$

$$= C \cos(\omega_0 t - \alpha)$$

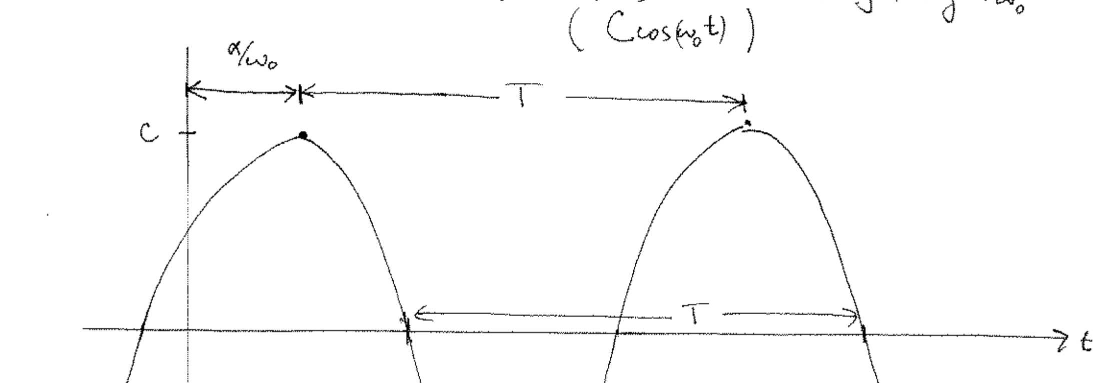
amplitude ↑ ↑ phase



(A, B) can be in any quadrant.

C = amplitude $= \sqrt{A^2+B^2}$
ω_0 = angular freq (rad/time)
$f = \omega_0/2\pi$ (cycles/sec. (hertz)) frequency
$T = 2\pi/\omega_0$ (time/cycles) period
α = phase angle;

since $\omega_0 t - \alpha = \omega_0(t - \alpha/\omega_0)$ $\quad \delta = \alpha/\omega_0$ is the time delay
this shifts standard cos curve to the right by α/ω_0
$(C\cos(\omega_0 t))$



Case 2 : Damping

$$m x'' + c x' + k x = 0$$

$$x'' + 2p x' + \omega_0^2 x = 0 \qquad \omega_0 = \sqrt{\tfrac{k}{m}} \text{ still}; \quad \tfrac{c}{m} = 2p; \quad p = \tfrac{c}{2m}$$

$$p(r) = r^2 + 2pr + \omega_0^2 = 0$$

$$r = \frac{-2p \pm \sqrt{4p^2 - 4\omega_0^2}}{2}$$

$$\boxed{r = -p \pm \sqrt{p^2 - \omega_0^2}}$$

Overdamped : $p^2 - \omega_0^2 > 0 \quad (c^2 > 4km)$

$$r = r_1 < r_2 < 0$$

$$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} = e^{r_1 t}\left(c_1 + c_2 e^{(r_2 - r_1)t}\right)$$

Figure 5.4.7 p 327

→ sol's decay exponentially to zero, cross t-axis at most 1 time.

Critically damped $p^2 - \omega_0^2 = 0 \quad (c^2 = 4km)$

$r = -p$ double root

$$x(t) = c_1 e^{rt} + c_2 t e^{rt} = e^{rt}(c_1 + c_2 t)$$

Figure 5.4.8 p 327 → sol's decay exponentially to zero, cross t-axis at most once

Underdamped $p^2 - \omega_0^2 < 0$

set $\omega_1 = \sqrt{\omega_0^2 - p^2} < \omega_0$

$$r = -p \pm i\omega_1$$

$$e^{rt} = e^{(-p \pm i\omega)t}$$

$$x(t) = e^{-pt}(A \cos\omega_1 t + B \sin\omega_1 t)$$

$$= e^{-pt}\left(C \cos(\omega_1 t - \alpha)\right)$$

Figure 5.4.9 p 328

pseudo period $T = \frac{2\pi}{\omega_1}$

pseudo freq $f = \frac{\omega_1}{2\pi}$

pseudo ang. freq ω_1

Ce^{-pt} : " time varying amplitude

1. solution oscillates, but oscillations are damped exponentially
2. motion is slowed relative to no damping ($\omega_1 < \omega_0$).

Exercise 1 On Monday we solved an undamped problem $c=0,\ m=2,\ k=18$

$$\begin{cases} x'' + 9x = 0 \\ x(0) = 1 \\ x'(0) = 3/2 \end{cases}$$

$$\begin{aligned} x_H(t) &= A\cos 3t + B\sin 3t \\ &= \cos 3t + .5\sin 3t \\ &= \frac{\sqrt{5}}{2}\cos(3t - .46) \qquad .46 = \arctan\left(\tfrac{.5}{1}\right) \\ &= \frac{\sqrt{5}}{2}\cos(3(t - .151)) \end{aligned}$$

$T = \frac{2\pi}{3} \approx 2.09$

Now add damping!

1a) $m=2,\ k=18,\ c=4$, same IV's

$$\begin{cases} x'' + 2x' + 9x = 0 \\ x(0)=1 \\ x'(0) = 3/2 \end{cases}$$

find $x_H(t)$, general sol'n
solve IVP
type of damping?
discuss

underdamped example

```
> restart:
> with(DEtools):
> Digits:=5:
> deqtn1:=diff(x(t),t,t)+2*diff(x(t),t)+9*x(t)=0:
> IC:=x(0)=1,D(x)(0)=1.5:
> dsolve({deqtn1,IC},x(t));
```

$$x(t) = \frac{5}{8}\sqrt{2}\,e^{(-t)}\sin(2\sqrt{2}\,t) + e^{(-t)}\cos(2\sqrt{2}\,t)$$

```
> evalf(%);
```

$$x(t) = 0.88388\,e^{(-1.\,t)}\sin(2.8284\,t) + e^{(-1.\,t)}\cos(2.8284\,t)$$

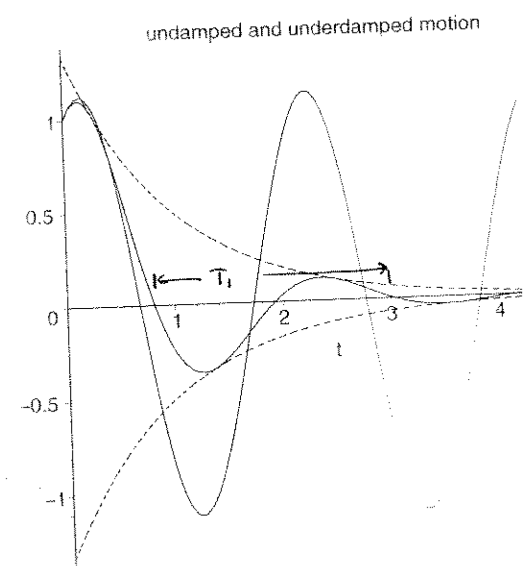
```
> x1:=t->.88388*exp(-1.*t)*sin(2.8284*t)+exp(-1.*t)*cos(2.8284*t):
> C1:=sqrt(.88388^2+1);  #pseudoamplitude
  omega1:=2.8284:   #slower (pseudo)-angular
                    #frequency than w/o damping
  T1:=evalf(2*Pi/omega1);  #longer "pseudoperiod"
  alpha1:=arctan(.88388); #pseudophase
```

$C1 := 1.3346$

$T1 := 2.2215$

$\alpha 1 := 0.72384$

```
> x2:=t->exp(-t)*C1*cos(omega1*t-alpha1): #same function
> with(plots):
> plot1:=plot(x1(t),t=0..5,color=black):
  plot2:=plot(x2(t),t=0..5,color=black):
  plot0:=plot(cos(3*t)+.5*sin(3*t),t=0..5,color=black):
     #undamped IVP solution
  plot3:=plot(C1*exp(-t),t=0..5,linestyle=2,color=black):
  plot4:=plot(-C1*exp(-t),t=0..5,linestyle=2,color=black):
    #the exponential "envelope", linestyle=2 means "dot"
  display({plot1,plot2,plot0,plot3,plot4},title=
    'undamped and underdamped motion');
```



1b) $\begin{cases} x'' + 6x' + 9x = 0 \\ x(0) = 1 \\ x'(0) = 3/2 \end{cases}$

Solve and discuss

1c) $\begin{cases} x'' + 10x' + 9x = 0 \\ x(0) = 1 \\ x'(0) = 3/2 \end{cases}$

Solve & discuss

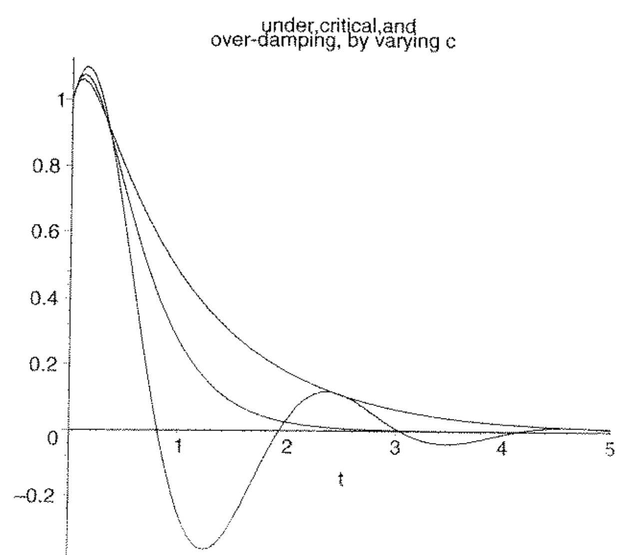
```
> deqtn2:=diff(x(t),t,t)+6*diff(x(t),t)+9*x(t)=0:
  IC:=x(0)=1,D(x)(0)=1.5:
  dsolve({deqtn2,IC},x(t)); #critical damping
```

$$x(t) = e^{(-3\,t)} + \frac{9}{2} e^{(-3\,t)}\, t$$

```
> deqtn3:=diff(x(t),t,t)+10*diff(x(t),t)+9*x(t)=0:
  IC:=x(0)=1,D(x)(0)=1.5:
  dsolve({deqtn3,IC},x(t)); #overdamping
```

$$x(t) = \frac{21}{16} e^{(-t)} - \frac{5}{16} e^{(-9\,t)}$$

```
> plot5:=plot(exp(-3*t)+9/2*exp(-3*t)*t,t=0..5,color=black):
  plot6:=plot(21/16*exp(-t)-5/16*exp(-9*t),t=0..5,color=black):
  display({plot5,plot6,plot2},title=`under,critical,and
  over-damping, by varying c`);
```



So much for unforced (homogeneous) spring problems. Now discuss general non-homogeneous linear DE, then in §5.6 we'll return to springs:

§5.5 : Finding y_p's for $\mathcal{L}(y)=f$.

Recall, if $\mathcal{L}$ is a linear operator the general sol'n to

$$\mathcal{L}(y)=f$$

is $\quad y = y_P + y_H$

where y_p is any particular sol'n and
y_H is the general (n-dim'l) solution to $\mathcal{L}(y)=0$

How to find y_p's.

Book calls the method "undetermined coefficients", for constant coefficient linear DE's, is really just "guessing" (but could be justified with vector space theory).

example 1

$$\mathcal{L}(y) = y'' + 3y' + 4y$$

solve $\mathcal{L}(y) = 3x+2$

try $y_p = Ax + B$ (because when you apply $\mathcal{L}$ to polys of degree ≤ 1 you get back polys of degree ≤ 1).

$$\begin{aligned} 4(\quad & y_p = Ax+B \\ +3(\quad & y_p' = A \\ +1(\quad & y_p'' = 0 \end{aligned}$$

$$\mathcal{L}(y_P) = x(4A) + 1(3A+4B) = 3x+2$$

$$\left.\begin{aligned} 4A &= 3 \\ 3A+4B &= 2 \end{aligned}\right\} \quad \begin{aligned} A &= 3/4 \\ \tfrac{9}{4} + 4B &= 2 \\ 4B &= -\tfrac{1}{4} \\ B &= -\tfrac{1}{16} \end{aligned}$$

$$\boxed{y_P = \frac{3}{4}x - \frac{1}{16}}$$

y_H: $\quad r^2 + 3r + 4 = 0$

$$r = \frac{-3 \pm \sqrt{9-16}}{2} = -\frac{3}{2} \pm \frac{i\sqrt{7}}{2}$$

$$\boxed{y_H(x) = e^{-3/2\,x}\left(A\cos\tfrac{\sqrt{7}}{2}x + B\sin\tfrac{\sqrt{7}}{2}x\right)}$$

$$\boxed{y(x) = \frac{3}{4}x - \frac{1}{16} + e^{-3/2\,x}\left(A\cos\tfrac{\sqrt{7}}{2}x + B\sin\tfrac{\sqrt{7}}{2}x\right)}$$

Tomorrow, we'll continue the "method of guessing" for right-hand sides of certain types. See table p.341
(poly, exps, trig, exp-trig)