Math 2250-1
Monday Oct 27
§5.3, 5.4

We're going through the algorithm to solve the constant coefficient linear homogeneous DE

$$\mathcal{L}(y) = 0$$

where $\mathcal{L}(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_2 y'' + a_1 y' + a_0 y$. $\quad a_j$'s constant.

in all cases, trying $y = e^{rx}$ leads to

$$\mathcal{L}(y) = (\underbrace{r^n + a_{n-1}r^{n-1} + \dots + a_1 r + a_0}_{p(r)})e^{rx}$$

so $\mathcal{L}(y) = 0$ exactly when r is a root of the charact poly $p(r)$.

We've understood
Case I : distinct real roots
Case II : repeated roots
and are discussing
Case III : complex roots.

$$\boxed{\text{Euler: } \quad e^{iy} := \cos y + i \sin y}$$

so, for $z = x + iy$, define

$$\boxed{e^z := e^x e^{iy} = e^x(\cos y + i \sin y)}$$

Def : If $f(x) + i g(x)$ is a complex-valued function of the real variable x,

then $\boxed{(f(x) + i g(x))' := f'(x) + i g'(x).}$

So if $\mathcal{L}$ is a linear differential operator with real coefficients, we can talk about complex solutions to

$$\mathcal{L}(y) = 0.$$

If $y = f + ig$, then $\mathcal{L}(y) = \mathcal{L}(f) + i\mathcal{L}(g)$, so $\mathcal{L}(y) = 0$ iff $\mathcal{L}(f) = 0$ and $\mathcal{L}(g) = 0$.

Theorem : If r is a complex root of the characteristic polynomial, then

$y = e^{rx}$ is a complex sol'n of $\mathcal{L}(y) = 0$.

why All we used to connect roots to solutions was $\boxed{\frac{d}{dx} e^{rx} = r e^{rx}}$

since then

$$\mathcal{L}(e^{rx}) = p(r)e^{rx}.$$

Check box on next page!

Let $r = a+bi$

then $e^{rx} = e^{(a+bi)x} = e^{ax}(\cos bx + i\sin bx)$

so $\frac{d}{dx}(e^{rx}) = \frac{d}{dx}(7_0) = e^{ax}\left[a(\cos bx + i\sin bx) + -b\sin bx + ib\cos bx\right]$

$= e^{ax}\left[a\cos bx - b\sin bx + i(a\sin bx + b\cos bx)\right]$

$\stackrel{?}{=} (a+bi)\, e^{ax}(\cos bx + i\sin bx)$ ✓.

$= re^{rx}$

Thus, if r is a root of the characteristic poly, then

$$\mathcal{L}(e^{rx}) = p(r)e^{rx} = 0 = 0 + 0i$$

$$\mathcal{L}(\underbrace{e^{ax}\cos bx}_{y_1(x)} + i\,\underbrace{e^{ax}\sin bx}_{y_2(x)})$$

$$\mathcal{L}(y_1(x)) + i\,\mathcal{L}(y_2(x))$$ by def of complex deriv.

Equating real & imag. parts, deduce

$$\mathcal{L}(y_1) = 0, \quad \mathcal{L}(y_2) = 0$$

gives 2 real linearly independent sol's,

$$e^{ax}\cos bx, \; e^{ax}\sin bx.$$

y_1, y_2 ind:

$c_1 e^{ax}\cos bx + c_2 e^{ax}\sin bx \equiv 0$

$\Rightarrow c_1\cos bx + c_2\sin bx \equiv 0$

$D_x \Rightarrow -c_1\sin bx\,(b) + c_2 b\cos bx \equiv 0$

@ $x=0$ get $c_1 + 0 = 0$

$0 + bc_2 = 0$

so $c_1 = c_2 = 0$

$\boxed{\text{Case III complex roots}}$

Each pair of conjugate roots $a+bi, a-bi$ yield a pair of linearly ind. sol'ns

$$e^{ax}\cos bx, \; e^{ax}\sin bx$$

If $r = a\pm bi$ appears with multiplicity >1 in the characteristic poly, you also get sol'ns

$$xe^{ax}\cos bx, \; xe^{ax}\sin bx$$

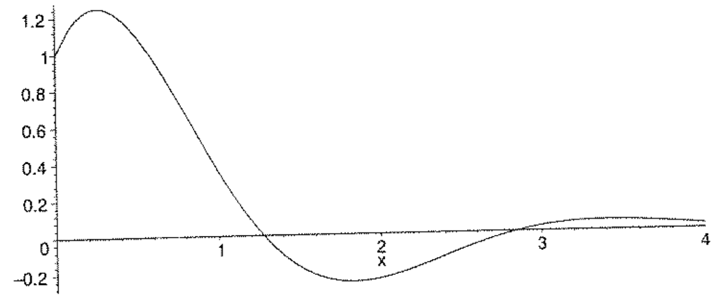
$$\vdots$$

following the same algorithm as for repeated real roots as in Case II.

example 1

$$\begin{cases} y'' + 2y' + 5 = 0 \\ y(0) = 1 \\ y'(0) = 2 \end{cases}$$

```
> with(plots):
Warning, the name changecoords has been redefined
> plot(exp(-x)*(cos(2*x)+1.5*sin(2*x)),x=0..4,color=black);
```



$p(r) = r^2 + 2r + 5 = 0$

$r = \frac{-2 \pm \sqrt{4-20}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$

$e^{(-1+2i)x} = e^{-x}(\cos 2x + i \sin 2x)$

$y(x) = c_1 e^{-x} \cos 2x + c_2 e^{-x} \sin 2x$

$y(0) = 1 = c_1$

$y'(0) = 2 = -c_1 + 2c_2$ ← why?

$2 = -1 + 2c_2$

$3/2 = c_2$

$$\boxed{y(x) = e^{-x}\left(\cos 2x + \tfrac{3}{2} \sin 2x\right)}$$

§5.4 Mechanical vibrations (= application, so $x(t)$ instead of $y(x)$).

$$\underbrace{m}_{\text{mass}} x'' + \underbrace{c}_{\text{coeff of friction}} x' + \underbrace{k}_{\text{Hooke's constant.}} x = 0$$

(see lecture notes 10/22 page 3 for derivation)

example 2

- interpret example 1 in this context.

We study this extremely important DE in stages:

Case 1 $\underbrace{\text{Free}}_{\text{no driving force}}$ $\underbrace{\text{undamped}}_{c=0}$ motion

$m x'' + k x = 0$

$x'' + \frac{k}{m} x = 0$

$x'' + \omega_0^2 x = 0 \qquad \left(\omega_0 := \sqrt{\frac{k}{m}}\right)$

if $x = e^{rt}$, $p(r) = r^2 + \omega_0^2 = 0$

$r = \pm i\omega_0$

$e^{i\omega_0 t} = \cos \omega_0 t + i \sin \omega_0 t$

$$\boxed{\begin{array}{l} x(t) = A \cos \omega_0 t + B \sin \omega_0 t \\ \text{simple harmonic motion} \end{array}}$$

example 3 A mass of 2 kg oscillates without damping on a spring with Hooke's constant 18 Newtons/meter. It is initially stretched 1 meter from equilibrium, and released with a velocity of 3/2 m/sec. Show that the displacement x(t) solves

A)
$$\begin{cases} x'' + 9x = 0 \\ x(0) = 1 \\ x'(0) = 3/2 \end{cases}$$

B) Solve the IVP (A). Express the solution in amplitude-phase form, and interpret the (t, x(t)) graph below (Identify amplitude, phase, time delay).

```
> with(plots):
  Digits:=5:
> omega:=3;                  #angular frequency
  alpha:=arctan(.5);         #phase
  delta:=arctan(.5)/3;       #time delay
  C:=sqrt(5./4);             #amplitude
  T:=evalf(2*Pi/3);          #period
```

0.46365

0.15455

1.1180

$T := 2.0944$

```
> plot1:=plot(cos(3*t)+.5*sin(3*t),t=0..4,color=black):
  plot2:=plot(C*cos(3*t-alpha),t=0..4,color=black):
  plot3:=plot(C*cos(3*(t-delta)),t=0..4,color=black):
  display({plot1,plot2,plot3});
```