Math 2250-1
Friday October 24
5.3 homog. DE's with const coeffs

HW for Friday Oct 31
5.2 2, 5, 9, 11, 13, 21, 22, 25, 26
5.3 3, 10, 14, 21, 22, 24, 29, 33, 37
5.4 4, 5, 7, 10, 12, 15, 17, 18, 23 (In 15, 17, 18 just find the solutions x(t), also in phase-amp. form)
5.5 3, 4, 12, 13, 19, 34, 37, 43, 49, 52
(Maple project on 5.4 & 5.6 is due Monday Nov. 3)

Recall general linear DE setup from Wednesday (& 5.2):

$$L(y) = y^{(n)} + p_{n-1}y^{(n-1)} + \cdots + p_2 y'' + p_1 y' + p_0 y \quad , \qquad p_j(x), \ f(x) \text{ continuous on } I$$

$L(y) = 0$ homogeneous

$L(y) = f$ inhomogeneous.

We know (pages 3-4 Wed, let's quickly rescan these)

- IVP $\begin{cases} L(y) = f \\ y(a) = b_0 \\ y'(a) = b_1 \\ \vdots \\ y^{(n-1)}(a) = b_{n-1} \end{cases}$ has unique solutions
- Solution space to $L(y) = 0$ is n-dimensional vector space.
- You can use the Wronskian to see if you've found a basis (how?)

And (discussed Tuesday but not written in the notes):

- The general solution to $L(y) = f$ is $y = y_P + y_H$ (y_P: partic soln; y_H: general soln to homog. eqtn.)
  Because
  (i) If $L(y_P) = f$ and $L(y_H) = 0$
  then $L(y_P + y_H) = L(y_P) + L(y_H) = f + 0 = f$
  so $y_P + y_H$ is always a soltn.
  (ii) If $L(y_P) = f$ and also $L(y) = f$
  then $L(y - y_P) = f - f = 0$
  so $y - y_P = y_H$ (a homog sol'n)
  $y = y_P + y_H$.

<u>Example 4</u> On Wednesday we showed that for

$$L(y) = y''' - 3y'' - 4y' + 12y$$

the soltns to $L(y) = 0$ are

$$y_H(x) = c_1 e^{2x} + c_2 e^{-2x} + c_3 e^{3x} \qquad (\{e^{2x}, e^{-2x}, e^{3x}\} \text{ are a basis for the sol'n space})$$

(y_H:) book writes $y_c(x)$, for "complementary sol'n".

Find the full solution to

$$y''' - 3y'' - 4y' + 12y = 6$$

hint: try a particular sol'n which is constant.

Example 5 Rework the chapter 1 problem

$$y' - 7y = 14$$

to find $y = y_p + y_H$. What is a basis (2 dimension) of the sol'n space to $\mathcal{L}(y) = 0$?

How to solve constant coefficient linear homogeneous DE's

$$\mathcal{L}(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_1 y' + a_0 y = 0 \qquad a_j\text{'s constant}$$

step 1 try $y = e^{rx}$.

Then $\mathcal{L}(y) = (\underbrace{r^n + a_{n-1}r^{n-1} + \dots + a_1 r + a_0}_{p(r),\ \text{the characteristic polynomial}})e^{rx}$

So any root of $p(r)$ yields a solution

Now there a several cases of increasing complexity.

Case I If $p(r)$ has n distinct (different) real roots $r_1, r_2, \dots r_n$ then

$$y_H(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \dots + c_n e^{r_n x}$$

is the general soltn (i.e. $\{e^{r_1x}, e^{r_2x}, \dots e^{r_nx}\}$ is a basis)

at $x=0$ the Wronskian matrix is

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ r_1 & r_2 & & r_n \\ r_1^2 & r_2^2 & & r_n^2 \\ \vdots & \vdots & & \vdots \\ r_1^{n-1} & r_2^{n-1} & & r_n^{n-1} \end{bmatrix}$$

with $\det = \prod_{i<j}(r_j - r_i) \neq 0!$, so these n fcns are lin ind, and basis!

(Example 4 Monday is this case)

Case II Repeated roots (real).

(We rushed an example of this on Wednesday)

If $(r-\alpha)^k$ is a factor of $p(r)$, then

- $e^{\alpha x}, xe^{\alpha x}, \dots x^{k-1}e^{\alpha x}$ are k linearly ind. soltns.
- If $p(r) = (r-r_1)^{k_1}(r-r_2)^{k_2}\cdots(r-r_j)^{k_j}$ $\quad k_1 + k_2 + \dots + k_j = n$ then a basis of soltns is given by

$$\{\underbrace{e^{r_1x}, xe^{r_1x}, \dots x^{k_1-1}e^{r_1x}}_{k_1\ \text{fcns}}, \underbrace{e^{r_2x}, xe^{r_2x}, \dots x^{k_2-1}e^{r_2x}}_{k_2\ \text{fcns}}, \dots \underbrace{e^{r_jx}, xe^{r_jx}, \dots x^{k_j-1}e^{r_jx}}_{k_j\ \text{fcns}}\}$$

one could check that these n functions are linearly independent (harder than case I), and that they all solve $\mathcal{L}(y) = 0$ (page 312).

<u>Example 6</u> $\quad y'''' - y''' = 0 \quad \longleftarrow$ This is an example of Case II, and it's easy enough that we can check ans by antidifferentiation:

$$p(r) = r^4 - r^3 = r^3(r-1)$$

so $\boxed{y_H(x) = \underset{\substack{\uparrow \\ 1 = e^{0x}}}{c_1} + c_2 x + c_3 x^2 + c_4 e^x}$

<u>Case III</u> Complex roots: exponential fcns still work, you just need to remember (or learn!) Euler's formula.

Recall from Taylor series in Calc.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \qquad = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

If we write $i = \sqrt{-1}$ then define

$$e^{ix} := 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \dots$$

$$= 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \frac{x^6}{6!} - \frac{ix^7}{7!} + \dots$$

$\cos x$!

$i \sin x$

$\boxed{e^{ix} := \cos x + i \sin x}$ Euler's formula

Define, for $z = x + iy$

$$\boxed{e^z = e^{x+iy} := e^x e^{iy} = e^x(\cos y + i \sin y)}$$

Now, return to the DE $\mathcal{L}(y) = 0$, with $r = a \pm ib$ roots of the characteristic polynomial. We claim

$$e^{(a+ib)x}, \quad e^{(a-ib)x} \quad \text{are (complex) solutions.}$$

$\uparrow$ $(f(x) + ig(x))' := f'(x) + ig'(x)$

$\Rightarrow \mathcal{L}(f + ig) = \mathcal{L}(f) + i\mathcal{L}(g)$ so if $y = f + ig$ is a complex solution,

this is equivalent to $\mathcal{L}(f) = 0 = \mathcal{L}(g)$

This follows from the fact that

$$\boxed{\frac{d}{dx} e^{(a+ib)x} = (a+ib)e^{(a+ib)x}}$$

$\uparrow$

a good exercise in trig identities, p. 315

Thus if $r = a \pm ib$ are conjugate roots of the characteristic poly $p(r) = 0$,

then box on left of bottom page 2 says

$$D_x e^{rx} = re^{rx}.$$

Thus $\mathcal{L}(e^{rx}) = p(r)e^{rx}$

$\equiv 0$ since $r = a \pm ib$ are roots.

On the other hand

$$e^{rx} = e^{ax}(\cos bx \pm i \sin bx)$$
$$= f + ig$$

so $\underset{0+0i}{\overset{\parallel}{0}} = \mathcal{L}(e^{rx}) = \underbrace{\mathcal{L}(e^{ax}\cos bx)}_{\text{real}} \pm i \underbrace{\mathcal{L}(e^{ax}\sin bx)}_{\text{imaginary}}.$

thus $\mathcal{L}(e^{ax}\cos bx) = 0$, $\mathcal{L}(e^{ax}\sin bx) = 0$. (2 sol'ns from two roots, $a \pm ib$).

lin ind: $c_1 e^{ax}\cos bx + c_2 e^{ax}\sin bx \equiv 0$

$\div e^{ax} \Rightarrow$ $c_1 \cos bx + c_2 \sin bx \equiv 0$

D_x: $-c_1 b \sin bx + c_2 b \cos bx \equiv 0$

@ $x=0$ yields $c_1 + 0 = 0$

$0 + bc_2 = 0$

$\Rightarrow c_1 = c_2 = 0.$

Example 7 Find the general sol'n to

$$y'' + 4y = 0$$

(notice this is our undamped spring from Tuesday notes, written with different letters).

$p(r) = r^2 + 4 = 0$

$r^2 = -4$

$r = \pm 2i$

$e^{(2i)x} = e^{i(2x)} = \cos 2x + i \sin 2x$

$y_H = c_1 \cos 2x + c_2 \sin 2x$ ✓

Case III completed: If $p(r) = (r-(a+ib))^k q(r)$ then $(r-(a-ib))^k$ is also a factor (since we assume $p(r)$ has real number coeff's)

and $e^{ax}\cos bx$, $e^{ax}\sin bx$

$xe^{ax}\cos bx$, $xe^{ax}\sin bx$

$\vdots$

$x^{k-1}e^{ax}\cos bx$, $x^{k-1}e^{ax}\sin bx$

are $2k$ linearly ind. sol'ns.

combining cases I, II, III yields an algorithm for solving $\mathcal{L}(y) = 0$ whenever $\mathcal{L}$ is an n^{th} order, const-coeff linear DE operator.

On Monday we will cover §5.4 – the mass-spring unforced oscillator

$$m x''(t) + c x'(t) + k x(t) = 0.$$

Look ahead!