

Math 2250-1

Wednesday Oct 22

§ 5.2-5.3 : begin with page 5 Tuesday. Then:

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Example 1 : Find all solutions to the 3rd order linear homogeneous differential equation

$$\mathcal{L}(y) = y'''(x) - 3y''(x) - 4y' + 12y = 0$$

let's try $y = e^{rx}$:

$$\begin{aligned} y' &= r e^{rx} \\ y'' &= r^2 e^{rx} \\ y''' &= r^3 e^{rx} \end{aligned}$$

$$\mathcal{L}(y) = (r^3 - 3r^2 - 4r + 12) e^{rx}$$

want roots of the characteristic polynomial

possible integer roots divide 12

$$\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12,$$

X ↑

$$r=2 : 8 - 12 - 8 + 12 = 0 \checkmark$$

deduce

$$\begin{aligned} y_1 &= e^{2x} \\ y_2 &= e^{-2x} \\ y_3 &= e^{3x} \end{aligned}$$

are solutions.

$$\begin{array}{r} r^2 - r - 6 \\ r-2 \overline{) r^3 - 3r^2 - 4r + 12} \\ \underline{r^3 - 2r^2} \\ -r^2 - 4r \\ \underline{-r^2 + 2r} \\ -6r + 12 \\ \underline{-6r + 12} \\ 0 \end{array}$$

\mathcal{L} is a linear operator, (see page 3)

$$p(r) = (r-2)(r^2 - r - 6) = (r-2)(r+2)(r-3)$$

So $\mathcal{L}(c_1 y_1 + c_2 y_2 + c_3 y_3) = \mathcal{L}(c_1 y_1) + \mathcal{L}(c_2 y_2) + \mathcal{L}(c_3 y_3)$
 $= c_1 \mathcal{L}(y_1) + c_2 \mathcal{L}(y_2) + c_3 \mathcal{L}(y_3)$
 $= 0$

So $c_1 y_1 + c_2 y_2 + c_3 y_3$ also satisfies this linear homogeneous DE.

In physics you call this the principle of superposition.

$$\left[\begin{array}{l} \mathcal{L}(y) = f \\ \mathcal{L}(z) = g \end{array} \right. \text{ then } \left. \begin{array}{l} \mathcal{L}(y+z) = f+g \\ \mathcal{L}(ay+bz) = af+bg \\ a, b \text{ const.} \end{array} \right]$$

So if V is our solution space to

$\mathcal{L}(y) = 0$, we see that any

$$c_1 y_1 + c_2 y_2 + c_3 y_3 \in V.$$

Is $\text{span}\{y_1, y_2, y_3\}$ all of V ?

Are y_1, y_2, y_3 linearly independent?

(continue on next page!)

Example 2: Solve the initial value problem:

$$\text{IVP } \begin{cases} y'''(x) - 3y''(x) - 4y'(x) + 12y(x) = 0 \\ y(0) = 0 \\ y'(0) = -3 \\ y''(0) = 5 \end{cases}$$

$$\begin{aligned} y_1 &= e^{2x} \\ y_2 &= e^{-2x} \\ y_3 &= e^{3x} \end{aligned}$$

$$\begin{aligned} y(x) &= c_1 y_1 + c_2 y_2 + c_3 y_3 \\ y'(x) &= c_1 y_1' + c_2 y_2' + c_3 y_3' \\ y''(x) &= c_1 y_1'' + c_2 y_2'' + c_3 y_3'' \end{aligned}$$

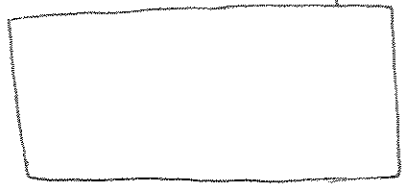
$$\begin{aligned} \begin{bmatrix} y \\ y' \\ y'' \end{bmatrix} &= \begin{bmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\ \begin{bmatrix} y \\ y' \\ y'' \end{bmatrix} &= \begin{bmatrix} e^{2x} & e^{-2x} & e^{3x} \\ 2e^{2x} & -2e^{-2x} & 3e^{3x} \\ 4e^{2x} & 4e^{-2x} & 9e^{3x} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \end{aligned}$$

If you have n solutions to a linear homogeneous DE, the matrix for which column j is $\begin{bmatrix} y_j \\ y_j' \\ \vdots \\ y_j^{(n-1)} \end{bmatrix}$ is called the Wronskian matrix. Its determinant is called the Wronskian $W(y_1, y_2, \dots, y_n)$

at $x=0$ this is the matrix eqn.

$$\begin{bmatrix} 0 \\ -3 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -2 & 3 \\ 4 & 4 & 9 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

Could we have solved any IVP for this DE, at $x=0$? Could the Wronskian explain why?



```
> A:=matrix(3,4,[1,1,1,0,
                2,-2,3,-3,
                4,4,9,5]);
A:=
[ 1  1  1  0
  2 -2  3 -3
  4  4  9  5]
> rref(A);
c1=-2
c2=1
c3=1
[ 1  0  0 -2
  0  1  0  1
  0  0  1  1]
```

$$y(x) = -2e^{2x} + e^{-2x} + e^{3x}$$

See theory pages 2 & 3. \rightarrow In particular, they span the solution space and are linearly independent. So they are a basis.

Theory pages

an n^{th} order differential operator

$$\mathcal{L}(y) = P_n(x)y^{(n)} + P_{n-1}(x)y^{(n-1)} + \dots + P_1(x)y'(x) + P_0(x)y(x)$$

is called linear, because

$$* \begin{cases} \mathcal{L}(y_1 + y_2) = \mathcal{L}(y_1) + \mathcal{L}(y_2) \\ \mathcal{L}(cy_1) = c\mathcal{L}(y_1) \end{cases}$$

The differential equation

$$\mathcal{L}(y) = F(x)$$

is homogeneous if $F(x) \equiv 0$. Otherwise it is inhomogeneous

Theorem 1 : The solution space to the linear, homogeneous, DE

$$\mathcal{L}(y) = 0$$

is a vector space, i.e. it is closed under addition and scalar multiplication.

Thus if y_1, \dots, y_n are solutions, so is

$$c_1y_1 + c_2y_2 + \dots + c_ny_n.$$

why: If y_1, y_2 are solutions, then $\mathcal{L}(y_1 + y_2) \stackrel{\text{by } *}{=} \mathcal{L}(y_1) + \mathcal{L}(y_2) = 0 + 0$
so $y_1 + y_2$ is.

If y_1 is a solution, then

$$\mathcal{L}(cy_1) = c\mathcal{L}(y_1) = 0, \text{ so } cy_1 \text{ is too.}$$

More generally,

$$\begin{aligned} \mathcal{L}(c_1y_1 + c_2y_2 + \dots + c_ny_n) &= \mathcal{L}(c_1y_1) + \mathcal{L}(c_2y_2 + \dots + c_ny_n) \\ &= c_1\underbrace{\mathcal{L}(y_1)}_0 + \underbrace{\mathcal{L}(c_2y_2 + \dots + c_ny_n)}_0 \\ &= \dots = 0 + 0 + \dots + 0 = 0. \end{aligned}$$

Theorem 2 : Let

$$\text{IVP } \begin{cases} \mathcal{L}(y) = y^{(n)} + p_{n-1}(x)y^{(n-1)} + \dots + p_0(x)y \equiv 0 \\ y(a) = b_0 \\ y'(a) = b_1 \\ \vdots \\ y^{(n-1)}(a) = b_{n-1} \end{cases}$$

If p_{n-1}, \dots, p_0 are continuous on an interval I and if $a \in I$.

Then there is always exactly 1 solution to this initial value problem.

why: makes intuitive sense.
more explanation later in course.

Theorem 3: Let $\mathcal{L}(y) = y^{(n)} + p_{n-1}y^{(n-1)} + \dots + p_1y' + p_0y$
 be as in theorem 2. Then the solution space to
 $\mathcal{L}(y) = 0$ has dimension n. Thus if we can
 find n linearly independent solutions they will span the solution
 space.

why: Using the existence-uniqueness result, Theorem 2, we can find
 (well, actually, there exist) y_1, \dots, y_n solving $\mathcal{L}(y_i) = 0$, with initial values

$$\begin{bmatrix} y_1(a) \\ y_1'(a) \\ \vdots \\ y_1^{(n-1)}(a) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \begin{bmatrix} y_2(a) \\ y_2'(a) \\ \vdots \\ y_2^{(n-1)}(a) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \dots \quad \begin{bmatrix} y_n(a) \\ y_n'(a) \\ \vdots \\ y_n^{(n-1)}(a) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

$\{y_1, \dots, y_n\}$ are a basis:

(a) span: let $y(x)$ solve $\mathcal{L}(y) = 0$. Then

$$\begin{aligned} y(a) &= b_0 \\ y'(a) &= b_1 \\ &\vdots \\ y^{(n-1)}(a) &= b_{n-1} \end{aligned}$$

some #'s b_0, \dots, b_{n-1}

compare $y(x)$ to $b_0y_1(x) + b_1y_2(x) + \dots + b_{n-1}y_n(x)$

these 2 solutions solve the same IVP, so they must be the same function, by Theorem 2.

i.e. $y(x) = b_0y_1 + b_1y_2 + \dots + b_{n-1}y_n$

(b) linearly ind: if then

$$\begin{aligned} c_1y_1 + c_2y_2 + \dots + c_ny_n &= 0 \\ c_1y_1' + c_2y_2' + \dots + c_ny_n' &= 0 \\ &\vdots \\ c_1y_n^{(n-1)} + \dots + c_ny_n^{(n-1)} &= 0 \end{aligned}$$

at $x=a$ deduce $\begin{aligned} c_1 &= 0 \\ c_2 &= 0 \\ &\vdots \\ c_n &= 0 \end{aligned}$ □

Theorem 4: Let $\mathcal{L}(y)$ be as in Theorems 2 & 3.
 Let $\{y_1, \dots, y_n\}$ be n solutions to $\mathcal{L}(y) = 0$ on the interval I.

Let $W = \det \begin{bmatrix} y_1 & y_2 & \dots & y_n \\ y_1' & y_2' & \dots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \dots & y_n^{(n-1)} \end{bmatrix}$ (the Wronskian).

Let $a \in I$. Then if $W(a) \neq 0$, y_1, \dots, y_n are a basis!
 (so you don't need to compute W at every x!)

why: $W(a) \neq 0$ means every IVP at a has a unique solution expressed as a linear combination of y_1, y_2, \dots, y_n .

Example 3 Find the solution space to

$$y'' - 6y' + 9 = 0$$

try $y = e^{rx}$
 $y' = r e^{rx}$
 $y'' = r^2 e^{rx}$

$$\mathcal{L}(y) = (r^2 - 6r + 9)e^{rx} = 0$$
$$(r-3)^2 = 0$$

$$y_1 = e^{3x}$$

Where's y_2 ?

trick: try $y_2 = x e^{3x}$
 $y_2' = (1+3x)e^{3x}$ (product rule)
 $y_2'' = (3+3(1+3x))e^{3x}$

$$\mathcal{L}(y_2) = e^{3x} \begin{bmatrix} x(9-18+9) \\ +1(9-6) \end{bmatrix} = 0!$$

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = \begin{bmatrix} e^{3x} & x e^{3x} \\ 3e^{3x} & e^{3x}(1+3x) \end{bmatrix}$$

$$W(0) = \det \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} = 1 \quad \text{so } \{y_1, y_2\} \text{ is a basis (see Theorem 4 p 9)}$$

$y(x) = c_1 e^{3x} + c_2 x e^{3x}$

(note, $W(x) = e^{9x}$ is never 0).

example 3 is typical of double roots. if $p(r) = (r-\alpha)^2 = r^2 - 2\alpha r + \alpha^2$,
for the 2nd order, linear, homogeneous, const-coeff DE

$$y'' - 2\alpha y' + \alpha^2 y = 0$$

then $y_1 = e^{\alpha x}$
 $y_2 = x e^{\alpha x}$ are a basis for the solution space

note $\left. \begin{array}{l} y_2 = x e^{\alpha x} \\ y_2' = e^{\alpha x} (\alpha x + 1) \\ y_2'' = e^{\alpha x} (\alpha + \alpha(\alpha x + 1)) \end{array} \right\} \begin{array}{l} \cdot \alpha^2 \\ \cdot -2\alpha \\ \cdot 1 \end{array}$

$$\mathcal{L}(y_2) = e^{\alpha x} \begin{bmatrix} x(\alpha^2 - 2\alpha^2 + \alpha^2) \\ +1(-2\alpha + 2\alpha) \end{bmatrix} = 0.$$