

Math 2250-1

Tuesday 10/21

§5.1 2nd order linear DE's.

p.1-2 theory } we'll discuss
p.3-4 examples } these in parallel.

Maple 2 project is posted!

We'll do the last page of Monday's notes at the end of class, if time.

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Def. A second order linear differential equation has the form

$$A(x)y'' + B(x)y' + C(x)y = F(x)$$

We search for solutions $y(x)$ on some interval $[a, b]$.

In this chapter we assume $A(x) \neq 0$ on the interval, so the DE could also be written (after dividing by $A(x)$)

$$y'' + p(x)y' + q(x)y = f(x)$$

One reason this differential eqn is called linear is that the "operator" \mathcal{L} defined by

$$\mathcal{L}(y) = y'' + p(x)y' + q(x)y$$

satisfies the linear operator axioms (like matrix mult axioms)

$$(1) \mathcal{L}(y_1 + y_2) = \mathcal{L}(y_1) + \mathcal{L}(y_2)$$

$$(2) \mathcal{L}(cy_1) = c\mathcal{L}(y_1)$$

[we checked these in detail for a particular example on Monday].

Therefore, by vector space theory, the solution set to the homogeneous DE

$$\mathcal{L}(y) = y'' + p(x)y' + q(x)y = 0$$

is a subspace. Add the general solution to $\mathcal{L}(y) = f$ is $y = y_p + y_H$, where y_p is any particular solution, and y_H is the general soltn to the homogeneous equation.

discussed in notes from 9/23, for first order linear DE.

Do you recall the reason??

Existence-Uniqueness theorem

If $p(x), q(x)$ are continuous on the interval I , and $a \in I$. Then there is a unique soltn to

$$\text{IVP } \begin{cases} y''(x) + p(x)y' + q(x)y = f(x) \\ y(a) = b_0 \\ y'(a) = b_1 \end{cases}$$

one reason this makes sense:

$$y'' = f - py' - qy$$

so if I know $y(a), y'(a)$, then I know $y''(a)$.

$$\text{But } y''' = f' - p'y' - py'' - q'y - qy'$$

so I can find $y'''(a)$.

so I can find all derivs of y at a , i.e. the Taylor series for y .

[of course this would require all fcn's to be infinitely differentiable.]

In theory, text uses
 $x = \text{variable}$
 $y(x) = \text{function}$

Consequence of the existence uniqueness theorem:

Theorem The solution space to the homogeneous DE

$$\mathcal{L}(y) = y'' + py' + qy = 0$$

is 2-dimensional. (So our goal is to find a basis consisting of 2 linearly ind. solutions)

proof: let $y_1(x)$ solve $\mathcal{L}(y_1) = 0$, $\begin{cases} y_1(a) = 1 \\ y_1'(a) = 0 \end{cases}$
 $y_2(x)$ solve $\mathcal{L}(y_2) = 0$, $\begin{cases} y_2(a) = 0 \\ y_2'(a) = 1 \end{cases}$

These solutions exist by existence-uniqueness theorem, and are unique

then the unique solution to

$$\begin{cases} y'' + py' + qy = 0 \\ y(a) = b_0 \\ y'(a) = b_1 \end{cases}$$

is $y(x) = b_0 y_1(x) + b_1 y_2(x)$

is a soltn ✓
 $y(a) = b_0 \cdot 1 + b_1 \cdot 0 = b_0$ ✓
 $y'(a) = b_0 \cdot 0 + b_1 \cdot 1 = b_1$ ✓

thus $\{y_1(x), y_2(x)\}$ are a basis for the solution space.

(they span by above.

they are independent because

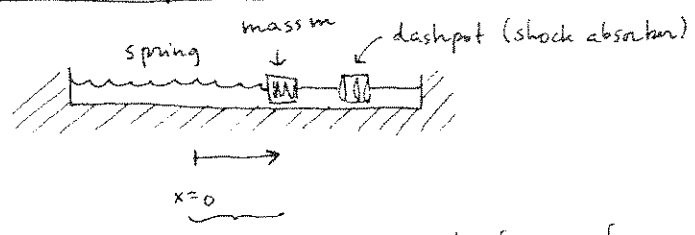
$$\begin{aligned} c_1 y_1 + c_2 y_2 &\equiv 0 \\ \Rightarrow c_1 y_1' + c_2 y_2' &\equiv 0 \end{aligned}$$

at $x=a$ we get

$$\begin{aligned} c_1 + 0 &= 0 \\ 0 + c_2 &= 0 \end{aligned} \quad \text{so } c_1 = c_2 = 0$$



Our most important example in the entire course



Model

Newton says

$$m x''(t) = \text{net force}$$

$$m x''(t) = F_s + F_d + F(t)$$

\uparrow \uparrow \uparrow
 spring drag any other
 force from applied
 dashpot forces.

Good models for spring and drag forces:

If $F_s = F_s(x)$ [only depends on displacement]
 then $F_s(x) = F_s(0) + F_s'(0)x + \frac{1}{2}F_s''(0)x^2 + \dots$ (Taylor series).
 • for small x ignore quadratic & above terms
 • $F_s(0) = 0$ since $x=0$ is equilibrium.
 $x'=0$

So $F_s \approx F_s'(0)x = -kx$ (coeff should be < 0 since)
 $x > 0 \Rightarrow F_s < 0$
 $x < 0 \Rightarrow F_s > 0$
Hooke's "Law"

If $F_d = F_d(v)$, where $v = x'(t)$
 then $F_d(v) \approx F_d(0) + F_d'(0)v + \dots$
 $F_d(v) \approx -cv$ linear drag

So $m x''(t) = -kx - cv + F(t)$ is a good model

IVP $\begin{cases} m x''(t) + c x'(t) + kx = F(t) \\ x(0) = x_0 \\ x'(0) = v_0 \end{cases}$

In applications
 text uses
 t = variable
 $x(t)$ = function

example In the mass-spring system suppose $m=2, k=8, c=0, F(t)=0$
 $\uparrow \quad \uparrow$
 $\text{kg} \quad \text{N/m}$

$$2x''(t) + 8x'(t) = 0$$

$$x'' + 4x = 0$$

$x_1(t) = \cos 2t$ are solutions, check!
 $x_2(t) = \sin 2t$

they are linearly ind., since if

$$c_1 x_1 + c_2 x_2 \equiv 0$$

$$\text{then } c_1 x_1' + c_2 x_2' \equiv 0$$

$$\begin{bmatrix} x_1 & x_2 \\ x_1' & x_2' \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \cos 2t & \sin 2t \\ -2\sin 2t & 2\cos 2t \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\text{at } t=0: \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow c_1 = c_2 = 0$$

this also shows we can solve any IVP:

$$\begin{cases} x'' + 4x = 0 \\ x(0) = 1 \\ x'(0) = -2 \end{cases}$$

$$\begin{aligned} c_1 x_1(0) + c_2 x_2(0) &= 1 \\ c_1 x_1'(0) + c_2 x_2'(0) &= -2 \end{aligned}$$

$$\begin{aligned} c_1 + 0 &= 1 \\ 0 + 2c_2 &= -2 \end{aligned}$$

$$\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 2 & -2 \\ \hline 1 & 0 & 1 \\ 0 & 1 & -1 \end{array}$$

$$\begin{aligned} c_1 &= 1 \\ c_2 &= -1 \end{aligned}$$

$$x(t) = \cos 2t - \sin 2t$$

Can you write $x(t)$ is phase-amplitude form?

how to find a basis for the sol'n space to
 $ay'' + by' + cy = 0$
 where $a \neq 0, b, c$ are constants.

try $y = e^{rx}$
 $y' = re^{rx}$
 $y'' = r^2 e^{rx}$

$\Rightarrow \mathcal{L}(y) = \underbrace{(ar^2 + br + c)}_{\text{set this poly in } r=0} e^{rx}$
 called characteristic polynomial

example

$y'' - 5y' + 6y = 0$

if $y = e^{rx}$
 get $(r^2 - 5r + 6)e^{rx} = 0$
 $(r-3)(r-2) = 0$

~~r=0,1~~ $r = 2, 3$

$y_1 = e^{2x}$
 $y_2 = e^{3x}$

$y(x) = c_1 e^{2x} + c_2 e^{3x}$

\rightarrow lin ind? $c_1 y_1 + c_2 y_2 = 0$
 $c_1 y_1' + c_2 y_2' = 0$

$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = 0$

$\begin{bmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{bmatrix}$

at $x=0 \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$; $\det = 3-2 = 1$
 so $c_1 = c_2 = 0$

In general,
 $W = \det \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}$ is called the Wronskian
 if it is $\neq 0$ anywhere,
 solutions are l.c.

Conclusion:
 $\{e^{2x}, e^{3x}\}$ is a basis
 for the 2-dim'l sol'n space

in an example
 $W = \begin{vmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{vmatrix} = e^{5x} \neq 0$ for all x .

