Math 2250-1
Monday Oct 20
4.4 & 4.7
How's your vocab?

4.1-4.3 HW due today!

HW for this Friday 10/24
4.4 1, 2, 3, 6, 8, 9, 11, 13, 26
4.7 1, 2, 3, 4, 5, 6, 7, 8, 9
10, 13, 15, 17, 22, 23, 25
5.1 2, 9, 11, 17, 27, 29, 30, 31,
34, 35, 39

V vector space
W subspace
<u>basis</u> for vectorspace (or subspace)
<u>dimension</u> of a vector space (or subspace)

Recall, the two easiest ways to get a subspace are as
- The solution set W to a homogeneous matrix equation $A\vec{x} = \vec{0}$
- the span of a collection of vectors, $W = \operatorname{span}\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_k\}$.

<u>Exercise 1</u> (from last Friday's notes):

Find a basis for the solutionspace to $A\vec{x} = \vec{0}$ for the matrix A shown at right. (backsolve from $\operatorname{rref}(A)|\vec{0}$.)

Explain why your basis <u>spans</u> W and why it's <u>linearly independent</u> (§4.4)

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> with(linalg):
> A:=matrix(4,6,[1,2,0,1,1,2,
                 2,4,1,4,1,7,
                 -1,-2,1,1,-2,1,
                 -2,-4,0,-2,-2,-4]);
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$$A := \begin{bmatrix} 1 & 2 & 0 & 1 & 1 & 2 \\ 2 & 4 & 1 & 4 & 1 & 7 \\ -1 & -2 & 1 & 1 & -2 & 1 \\ -2 & -4 & 0 & -2 & -2 & -4 \end{bmatrix} \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \end{matrix}$$

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> rref(A);
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$$\begin{bmatrix} 1 & 2 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \end{matrix}$$

Exercise 2 : Based on exercise 1,

2a) Why does rref & backsolve with free parameters always yield a basis for the solution space to $A\vec{x} = \vec{0}$?

span:

lin ind:

2b) What is the dimension of the solution space to $A\vec{x} = \vec{0}$, i.e. how can you tell what it is by looking at rref(A)?

And now for something completely different, yet completely the same! (This is §4.7)

Exercise 3 Recall the vector space $\mathcal{F} = \{$continuous functions defined on $\mathbb{R}\}$.

let $L(y) := y'''$.

let $W =$ the solution space to $L(y) = 0$

3a) Verify $(\alpha), (\beta)$ to show W is a subspace

α) $y_1, y_2 \in W \Rightarrow y_1 + y_2 \in W$:

β) $y \in W$, $k \in \mathbb{R} \Rightarrow ky \in W$:

3b) "solve" $y''' = 0$ (by antidifferentiation). Show W is 3-dimensional and exhibit a basis

Key facts about independence, span, basis, dimension

(in a fixed vector space V)

(1) **Primary fact**: If a finite collection of vectors spans a vector space V, then any collection having a greater number of vectors is **dependent**

reason: Let $\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_k\}$ span V, Let $\{\vec{w}_1, \vec{w}_2, \ldots \vec{w}_{k+\ell}\}$ $(\ell > 0)$ be a larger collection

We search for lin. comb. coeff's c_j s.t.

$$c_1\vec{w}_1 + c_2\vec{w}_2 + \cdots + c_{k+\ell}\vec{w}_{k+\ell} = \vec{0}$$

Express each $\vec{w}_j$ as a linear combo of the spanning set:

$$c_1\begin{bmatrix} a_{11}\vec{v}_1 \\ +a_{21}\vec{v}_2 \\ + \\ \vdots \\ +a_{k1}\vec{v}_k \end{bmatrix} + c_2\begin{bmatrix} a_{12}\vec{v}_1 \\ +a_{22}\vec{v}_2 \\ \vdots \\ +a_{k2}\vec{v}_k \end{bmatrix} + \cdots + c_{k+\ell}\begin{bmatrix} a_{1\,k+\ell}\vec{v}_1 \\ +a_{2\,k+\ell}\vec{v}_2 \\ \vdots \\ +a_{k\,k+\ell}\vec{v}_k \end{bmatrix} = \begin{bmatrix} 0\vec{v}_1 \\ +0\vec{v}_2 \\ + \vdots \\ +0\vec{v}_k \end{bmatrix}$$

equating coeff's for each $\vec{v}_i$, we get a dependency if we can find a $\vec{c} \neq \vec{0}$ so that

$$\begin{bmatrix} & A & \end{bmatrix}\begin{bmatrix} \\ \vec{c} \\ \\ \end{bmatrix} = \begin{bmatrix} \vec{0} \end{bmatrix}$$

But the solution space to this homogeneous equation is at least ℓ-dimensional, so non-zero sol'ns $\vec{c}$ exist. ∎

Logical consequences of (1):

(2) If a finite collection of vectors in V is independent, no collection with fewer vectors can span

reason: logic! If a smaller set did span, the larger set would've been dependent! ∎

(3) If $\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_k\}$ is a basis for V, then every other basis also consists of exactly k vectors ∎

reason: fewer vectors can't span, more vectors would be dependent.

(4) If $\dim V = k$ and if $\{\vec{w}_1, \vec{w}_2, \ldots \vec{w}_k\}$ are independent, they span! (and are a basis)

reason: if they didn't span there would be a $\vec{v} \in V$ not in their span, so that the set $\{\vec{w}_1, \vec{w}_2, \ldots \vec{w}_k, \vec{v}\}$ would still be independent. This would violate (1)! ∎

(5) If $\dim V = k$ and if $\{\vec{w}_1, \vec{w}_2, \ldots \vec{w}_k\}$ span, then they're independent! (so are a basis)

reason: if they were dependent we could throw one of them away (i.e. a dependent one) without shrinking the span, so we would have $(k-1)$ vectors spanning V, in violation of (2)! ∎

Exercise 4 Linear algebra explains why partial fractions always works! (§4.7)

For example, to solve

$$\frac{3x+4}{(x-2)(x+2)(x-1)} = \frac{A}{x-2} + \frac{B}{x+2} + \frac{C}{x-1}$$

$$= \frac{A(x+2)(x-1) + B(x-2)(x-1) + C(x-2)(x+2)}{(x-2)(x+2)(x-1)}$$

we set numerators equal:

$$3x+4 = A(x+2)(x-1) + B(x-2)(x-1) + C(x-2)(x+2)$$

unique soltn (A, B, C) exists iff

$\{(x+2)(x-1), (x-2)(x-1), (x-2)(x+2)\}$ are a basis for $\mathcal{P}_2$

by previous page, just need to check independence, since $\dim \mathcal{P}_2 = 3$.

but if $c_1(x+2)(x-1) + c_2(x-2)(x-1) + c_3(x-2)(x+2) \equiv 0$

(means LHS = 0 for all x);

@ $x=1 \Rightarrow c_3 = 0$

@ $x=-2 \Rightarrow c_2 = 0$ $\Rightarrow$ independent

@ $x=2 \Rightarrow c_1 = 0$ $\Rightarrow$ basis by 4 on page 3.

Thus unique A, B, C will exist.

- this can be generalized to explain why all the partial fraction "magic" always works!

4a) Find A, B, C, by plugging in good x-values

4b) Find A, B, C, by "collecting coefficients". (This amounts to expressing equations using the basis $\{1, x, x^2\}$ for $\mathcal{P}_2$)