Math 2250-3
Friday Oct 10

How's your vocabulary?

linear combination of $\vec{v}_1, \dots \vec{v}_k$

span $\{\vec{v}_1, \dots \vec{v}_k\}$

$\{\vec{v}_1, \dots \vec{v}_k\}$ linearly independent
linearly dependent

V a vector space

W a subspace [not just any subset!]

Recall, we postponed §4.1-4.3 HW 'til Oct 20 (Monday after break.)

That Friday, Oct 24, has this HW due:
(and on this set, like the last one, you may use techology to compute rref, once you've set up the problem.)

4.4 1, 2, 3, 6, 8, 9, 11, 13, 26
4.5 1, 12, 16, 17, 26, 27, 28
4.7 1, 2, 3, 4, 5, 6, 7, 8, 9,
10, 13, 15, 17, 21, 23, 25

§4.4

New

Def $\{\vec{v}_1, \dots \vec{v}_k\}$ is a basis for V iff
(a) $V = \text{span}\{\vec{v}_1, \dots \vec{v}_k\}$
(b) $\{\vec{v}_1, \dots \vec{v}_k\}$ linearly independent

this is equivalent to saying that each $\vec{v} \in V$ can be uniquely expressed as
$$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k$$
in this case the linear combo coefficients are called the coords of $\vec{v}$ with respect to the basis $\{\vec{v}_1, \dots \vec{v}_k\}$.

Example The standard basis $\{\vec{e}_1, \dots \vec{e}_n\}$ of $\mathbb{R}^n$

where $\vec{e}_j = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow$ entry j

(a) span: $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\vec{e}_1 + x_2\vec{e}_2 + \dots + x_n\vec{e}_n$

(b) linearly independent: if $c_1\vec{e}_1 + c_2\vec{e}_2 + \dots + c_n\vec{e}_n = \vec{0}$
then $\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ so all c_j's $= 0$

Exercise 1: On page 1 Wednesday, did we find a basis for $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \text{ s.t. } x + 2y - 3z = 0 \right\}$?
check!

after answering that question, finish Wednesday notes (Exercises 2-3, page 3 Example.)
Then return to today's notes ...

Recall, we figured out that

(a) more than n vectors in $\mathbb{R}^n$ are always linearly dependent

(b) less than n vectors in $\mathbb{R}^n$ cannot span $\mathbb{R}^n$

- so each basis of $\mathbb{R}^n$ has exactly n vectors

  and $\{\vec{v}_1, \dots \vec{v}_n\}$ is a basis for $\mathbb{R}^n$ iff

$$A = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix} \quad \text{satisfies } \operatorname{rref}(A) = I$$

Def For any vector space V, the dimension of V $(\dim(V))$ is defined ~~the~~ to be the number of vectors in every basis of V.

This definition only makes sense because

Theorem Every basis of V has the same number of vectors.

Lemma: Let $\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_k\}$ span V. Then any collection $\{\vec{w}_1, \vec{w}_2, \dots \vec{w}_{k+\ell}\}$ $(\ell > 0)$ of more than k vectors in V is linearly dependent.

proof of Lemma:

we search for lin combo coef's s.t.

$$c_1 \vec{w}_1 + c_2 \vec{w}_2 + \dots + c_{k+\ell} \vec{w}_{k+\ell} = 0.$$

Express each $\vec{w}_j$ as a linear combo of the v_i's:

$$c_1 \begin{bmatrix} a_{11}\vec{v}_1 \\ + a_{21}\vec{v}_2 \\ + \\ \vdots \\ + a_{k1}\vec{v}_k \end{bmatrix} + c_2 \begin{bmatrix} a_{12}\vec{v}_1 \\ + a_{22}\vec{v}_2 \\ + \\ \vdots \\ + a_{k2}\vec{v}_k \end{bmatrix} + \dots + c_{k+\ell} \begin{bmatrix} a_{1\,k+\ell}\vec{v}_1 \\ + a_{2\,k+\ell}\vec{v}_2 \\ + \\ \vdots \\ + a_{k,k+\ell}\vec{v}_k \end{bmatrix} = \begin{bmatrix} 0\vec{v}_1 \\ + 0\vec{v}_2 \\ + \\ \vdots \\ + 0\vec{v}_k \end{bmatrix}$$

We get a dependency if we can find $\vec{c} \neq \vec{0}$ so that

$$\begin{bmatrix} & A & \end{bmatrix} \begin{bmatrix} \\ \vec{c} \\ \\ \end{bmatrix} = \begin{bmatrix} \vec{0} \end{bmatrix}$$

But we can always do this because A has more columns than rows!

Lemma proven! ∎

proof of theorem: Let $\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_k\}$, $\{\vec{w}_1, \vec{w}_2, \dots \vec{w}_{k+\ell}\}$ $\ell \geq 0$ be two bases of V, where the second collection has at least as many elements as the first

If $\ell \geq 1$, then the $\vec{w}_j$'s would be dependent, i.e. not actually a basis.

Thus $\ell = 0$. ∎

Exercise 2 Find a basis for the solution space to $A\vec{x} = \vec{0}$ for the matrix A below. Make sure to explain why your basis spans the solution space, and why it's linearly independent

```
> with(linalg):
> A:=matrix(4,6,[1,2,0,1,1,2,
              2,4,1,4,1,7,
             -1,-2,1,1,-2,1,
             -2,-4,0,-2,-2,-4]);
```

homogeneous eq'n

$$A := \begin{bmatrix} 1 & 2 & 0 & 1 & 1 & 2 \\ 2 & 4 & 1 & 4 & 1 & 7 \\ -1 & -2 & 1 & 1 & -2 & 1 \\ -2 & -4 & 0 & -2 & -2 & -4 \end{bmatrix} \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \end{matrix}$$

```
> rref(A);
```

$$\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{array}$$
$$\begin{bmatrix} 1 & 2 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \end{matrix}$$

Exercise 3 : Based on exercise 2,

3a) Why does our rref & backsolving with free parameters always yield a basis for the solution space to $A\vec{x} = \vec{0}$? (As happened in Exercises 1, 2)

span:

lin ind:

3b) What is dimension of solution space to $A\vec{x} = \vec{0}$ (i.e. how can you calculate the dimension from rref(A)?