Math 2250-1
Wednesday October 1

Exam 1 Monday!
1.1–1.6
2.1–2.3
3.1–3.6

Review session
Saturday?
or
Friday afternoon?

(postpone §4.1 HW until next Friday.)

Finish determinants! §3.6; begin §4.1

Did: A^{-1} exists iff $\mathrm{rref}(A) = I$ iff $\det A \neq 0$

Theorem Recall the cofactor $C_{ij} = (-1)^{i+j} M_{ij}$
where the minor M_{ij} is the det of the $(n-1)\times(n-1)$ matrix obtained by deleting row i and col j of A

the adjoint matrix is the transpose of the cofactor matrix, $[C_{ij}] = \mathrm{cof}(A)$.
i.e.
$$\mathrm{Adj}(A) = \mathrm{cof}(A)^T$$
then, when A^{-1} exists, it has formula
$$A^{-1} = \frac{1}{|A|}\,\mathrm{adj}(A)$$

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exercise 1: Show that for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ this theorem reproduces our "magic" formula
$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

exercise 2 for our friend $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{bmatrix}$ we worked out $\mathrm{cof}(A) = \begin{bmatrix} 5 & 2 & -6 \\ 0 & 3 & 6 \\ 5 & -1 & 3 \end{bmatrix}$, $|A| = 15$

What is A^{-1}?
Check ans!

$\mathrm{adj}(A) = \begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix}$

exercise 3

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{bmatrix} \qquad \operatorname{cof}(A) = \begin{bmatrix} 5 & 2 & -6 \\ 0 & 3 & 6 \\ 5 & -1 & 3 \end{bmatrix}$$

$$\operatorname{adj}(A) = \begin{bmatrix} 5 & 0 & 5 \\ 2 & 3 & -1 \\ -6 & 6 & 3 \end{bmatrix}$$

3a) the (1,1) entry of

$[A][\operatorname{adj}(A)]$ is $15. = 1\cdot 5 + 2\cdot 2 - 1(-6)$

explain why this is det(A), expanded across first row

3b) the (2,1) entry of $[A][\operatorname{adj}(A)]$ is $0\cdot 5 + 3\cdot 2 + 1(-6) = 0$.

Notice, you used the same cofactors as in 3a).
What matrix (which uses 2 rows of A) is this the determinant of?

3c) the (3,2) entry of $[A][\operatorname{adj}(A)]$ is $2\cdot 0 + (-2)(3) + 1(6) = 0$

What matrix (which uses 2 rows of A) is this the det of?

If you understand 3a, b, c, completely, you have just realized why

$(A)(\operatorname{adj}(A)) = \det(A)\, I$ for every square matrix,

and so, why $A^{-1} = \frac{1}{\det A} \operatorname{adj}(A)$ when $\det A \neq 0$.

precisely, $\text{entry}_{ii}\, A(\operatorname{adj}(A)) = \sum_{j=1}^{n} a_{ij} C_{ij} = |A|$

$k \neq i$, $\text{entry}_{ik}\, A(\operatorname{adj}(A)) = \sum_{j=1}^{n} a_{ij} C_{kj} = \begin{vmatrix} \text{row}_1(A) \\ \vdots \\ \text{row}_i(A) \\ \vdots \\ \text{row}_i(A) \\ \vdots \\ \text{row}_n(A) \end{vmatrix} \begin{matrix} \\ \\ \leftarrow \text{slot } k \\ \\ \leftarrow \text{slot } i \\ \\ \\ \end{matrix} = 0.$

$\boxed{\text{this proves the } A^{-1} \text{ formula theorem!}}$

Cramer's Rule (see below) (complete proof in yesterday's notes)

the sol'n to

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 4 \end{bmatrix} \qquad A\vec{x} = \vec{b}$$

is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{15}\begin{bmatrix} 5 & 0 & 5 \\ 2 & 3 & -1 \\ -6 & 6 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 5 \\ 4 \end{bmatrix} =$

interpret with cofactors

1st col cofactors

$$x = \frac{1}{15}(5\cdot 2 + 0\cdot 5 + 5\cdot 4)$$

RHS

$$= \frac{\begin{vmatrix} 2 & 2 & -1 \\ 5 & 3 & 1 \\ 4 & -2 & 1 \end{vmatrix}}{15} = \frac{\left| \vec{b} \,\vdots\, col_2(A) \,\vdots\, col_3(A) \right|}{|A|}$$

2nd col cofactors

$$y = \frac{1}{15}(2\cdot 2 + 3\cdot 5 - 1\cdot 4)$$

$$= \frac{\begin{vmatrix} 1 & 2 & -1 \\ 0 & 5 & 1 \\ 2 & 4 & 1 \end{vmatrix}}{15} = \frac{\left| col_1(A) \,\vdots\, \vec{b} \,\vdots\, col_3(A) \right|}{|A|}$$

$z =$

If A^{-1} exists, and if $A\vec{x} = \vec{b}$, $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, then $\boxed{x_k = \frac{\left| col_1(A) \,\vdots\, col_2(A) \,\vdots\, \cdots \,\vdots\, \vec{b} \,\vdots\, \cdots \,\vdots\, col_n(A) \right|}{|A|}}$

i.e. the numerator is det of mat. obtained by replacing k^{th} column of A with $\vec{b}$.

exercise 4 Solve $\begin{bmatrix} 5 & -1 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$

a) with Cramer's rule

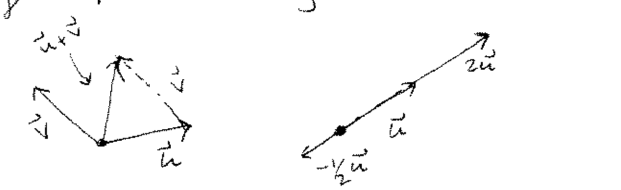
b) with A^{-1}, using adjoint formula

Introduction to § 4.1–4.3
(to be continued)

In chapter 4 we will continuously make use of the "linear combination form" way of writing a matrix times a vector:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & \cdots & & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \overset{\text{chapter 3}}{=} \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + \dots + a_{mn}x_n \end{bmatrix} \overset{\text{chapter 4}}{=} x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \dots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

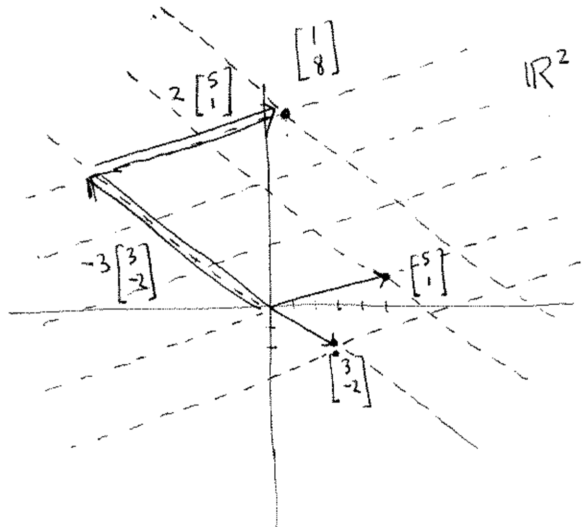
and you will need to remember the geometric meaning of vector addition and scalar multiplication.



<u>example</u>

How can you get to $\begin{bmatrix} 1 \\ 8 \end{bmatrix}$ in $\mathbb{R}^2$

by only moving in the $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} 5 \\ 1 \end{bmatrix}$ directions?



Solve

$$s\begin{bmatrix} 3 \\ -2 \end{bmatrix} + t\begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 5 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} s \\ t \end{bmatrix} = \frac{1}{13}\begin{bmatrix} 1 & -5 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 8 \end{bmatrix}$$

$$= \frac{1}{13}\begin{bmatrix} -39 \\ 26 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

the expression $s\begin{bmatrix} 3 \\ -2 \end{bmatrix} + t\begin{bmatrix} 5 \\ 1 \end{bmatrix}$ is called a <u>linear combination</u> of $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} 5 \\ 1 \end{bmatrix}$

- Can we get anywhere in $\mathbb{R}^2$ via a linear combination of $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} 5 \\ 1 \end{bmatrix}$?
- Are the linear combination coefficients (s & t) unique, if $s\begin{bmatrix} 3 \\ -2 \end{bmatrix} + t\begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$?

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example :

$\vec{i} = \vec{e}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\vec{j} = \vec{e}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$, $\vec{k} = \begin{bmatrix}0\\0\\1\end{bmatrix} = \vec{e}_3$

these 3 vectors are often called the standard basis vectors for $\mathbb{R}^3$

~~exam~~

$$\begin{bmatrix}7\\-1\\3\end{bmatrix} = 7\vec{e}_1 - \vec{e}_2 + 3\vec{e}_3$$

- Can we express any vector in $\mathbb{R}^3$ as a linear combination $c_1\vec{e}_1 + c_2\vec{e}_2 + c_3\vec{e}_3$ of $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$?
- Is this expression unique? (i.e. is there exactly 1 such expression?)

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example :

$\vec{v}_1 = \begin{bmatrix}2\\1\\1\end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix}-1\\1\\-5\end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix}4\\-1\\11\end{bmatrix}$

- is $\begin{bmatrix}6\\3\\3\end{bmatrix}$ a linear combination of $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$?
- is this expression (i.e. the linear combo coefficients) unique?

$$c_1\begin{bmatrix}2\\1\\1\end{bmatrix} + c_2\begin{bmatrix}-1\\1\\-5\end{bmatrix} + c_3\begin{bmatrix}4\\-1\\11\end{bmatrix} = \begin{bmatrix}6\\3\\3\end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 2 & -1 & 4 & 6\\ 1 & 1 & -1 & 3\\ 1 & -5 & 11 & 3 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 3\\ 0 & 1 & -2 & 0\\ 0 & 0 & 0 & 0 \end{array}\right]$$

$c_3 = t$
$c_2 = 2t$
$c_1 = 3 - t$; $\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}3\\0\\0\end{bmatrix} + t\begin{bmatrix}-1\\2\\1\end{bmatrix}$

so, e.g.

$\begin{bmatrix}6\\3\\3\end{bmatrix} = 3\vec{v}_1$

or $\begin{bmatrix}6\\3\\3\end{bmatrix} = 2\vec{v}_1 + 2\vec{v}_2 + \vec{v}_3$ $(t=1)$

or $\begin{bmatrix}6\\3\\3\end{bmatrix} = (3-t)\vec{v}_1 + 2t\vec{v}_2 + t\vec{v}_3$ any t

- Can we express any point in $\mathbb{R}^3$ as a linear combo of $\vec{v}_1, \vec{v}_2, \vec{v}_3$?
- If $\vec{b}$ is a linear combo of $\vec{v}_1, \vec{v}_2, \vec{v}_3$, are the linear combo coefficients unique?