Math 2250-1
Friday Nov. 7

next week's hw is in Wed notes!

More Laplace transform magic, §5.2-5.3

$$\mathcal{L}\{f(t)\}(s) := \int_0^\infty e^{-st} f(t)\,dt \quad = F(s)$$

for $s > M$, with $|f(t)| \le Ce^{Mt}$, f defined for $t \geq 0$

$\mathcal{L}$ is linear $\left(\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\}(s) = c_1 F_1(s) + c_2 F_2(s) \right)$

$\mathcal{L}$ is invertible (inverse Laplace transform yields a function defined for $t \geq 0$)

We're filling in a table of Laplace transforms, and using the table and algebra to solve IVP's.

warmup exercise

$$\mathcal{L}\{7 - 5\cos 3t + 2\sin 8t\}(s) =$$

$$\mathcal{L}^{-1}\left\{\frac{2}{s+3} + \frac{4}{s^2+9}\right\} =$$

(1b) use $\cosh(kt) := \frac{1}{2}\left(e^{kt} + e^{-kt}\right)$

$\sinh(kt) = \frac{1}{2}\left(e^{kt} - e^{-kt}\right)$

(2a) We did $f'(t)$, $f''(t)$ yesterday.
Check $f'''(t) = (f''(t))'$ to see pattern!

	$f(t)$	$F(s)$
	1	$\frac{1}{s}$
	e^{at}	$\frac{1}{s-a}$
	e^{ikt}	$\frac{1}{s-ik} = \frac{s+ik}{s^2+k^2}$
	$c_1 f_1(t) + c_2 f_2(t)$	$c_1 F_1(s) + c_2 F_2(s)$
1a	$\cos kt$	$\frac{s}{s^2+k^2}$
	$\sin kt$	$\frac{k}{s^2+k^2}$
1b	$\cosh kt$	$\frac{s}{s^2-k^2}$
	$\sinh kt$	$\frac{k}{s^2-k^2}$
2a	$f'(t)$	$sF(s) - f(0)$
	$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
	$f'''(t)$	$s^3F(s) - s^2f(0) - sf'(0) - f''(0)$
	etc.	
2b	$\int_0^t f(\tau)\,d\tau$	$\frac{F(s)}{s}$
2c	$t f(t)$	$-F'(s)$
	$t^2 f(t)$	$F''(s)$
	etc.	
3	t	$\frac{1}{s^2}$
	t^2	$\frac{2}{s^3}$
	t^n	$n!/s^{n+1}$, n int. ≥ 0
4a	$e^{at}f(t)$	$F(s-a)$
	$e^{at}\cos kt$	$\frac{s-a}{(s-a)^2+k^2}$
	$e^{at}\sin kt$	$\frac{k}{(s-a)^2+k^2}$
4b	$u(t-a)$	e^{-as}/s
	$u(t-a)f(t-a)$	$e^{-as}F(s)$

examples

te^{at}	$\frac{1}{(s-a)^2}$
$t\cos kt$	$-D_s\left(\frac{s}{s^2+k^2}\right)$
$t\sin kt$	$-D_s\left(\frac{k}{s^2+k^2}\right)$

2b let $g(t) = \int_0^t f(\tau)\,d\tau$

notice $g'(t) = f(t)$ (g is an antideriv of f)

$g(0) = 0$

so $\mathcal{L}\{g'(t)\}(s) = s\,G(s) - g(0)$

$F(s) = s\,G(s)$

$\frac{F(s)}{s} = G(s)$

2c

$$F'(s) = \lim_{\Delta s \to 0} \frac{1}{\Delta s}\left(F(s+\Delta s) - F(s)\right)$$

$$= \lim_{\Delta s \to 0} \underbrace{\frac{1}{\Delta s}\int_0^\infty e^{-(s+\Delta s)t} f(t) - e^{-st} f(t)\,dt}_{\int_0^\infty f(t)\left[\frac{e^{-(s+\Delta s)t} - e^{-st}}{\Delta s}\right]dt}$$

$$\frac{d}{ds} e^{-st} = -t e^{-st}$$

$$= \int_0^\infty -t f(t) e^{-st}\,dt = \mathcal{L}\{-t f(t)\}(s)$$

Thus $\mathcal{L}\{t f(t)\}(s) = -F'(s)$

then $\mathcal{L}\{t(t f(t))\}(s) = -(-F''(s))$ etc.

3 Use **2c**: $\mathcal{L}\{1\}(s) = \frac{1}{s}$ (yesterday)

$\mathcal{L}\{t \cdot 1\}(s) = -\left(\frac{1}{s}\right)' = \frac{1}{s^2}$

$\mathcal{L}\{t \cdot t\}(s) = -\left(\frac{1}{s^2}\right)' = \frac{2}{s^3}$

$\mathcal{L}\{t \cdot t^2\}(s) = -\left(\frac{2}{s^3}\right)' = \frac{3!}{s^4}$ etc!

4a $\mathcal{L}\{e^{at} f(t)\}(s) = \int_0^\infty e^{-st} e^{at} f(t)\,dt = \int_0^\infty e^{-(s-a)t} f(t)\,dt = F(s-a)$!

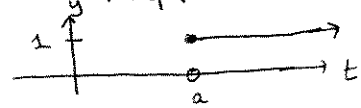
thus $\mathcal{L}\{e^{at}\cosh kt\}(s) = \mathcal{L}\{\cosh kt\}(s-a) = \frac{s-a}{(s-a)^2 + k^2}$

$\mathcal{L}\{e^{at}\sinh kt\}(s) = \mathcal{L}\{\sinh kt\}(s-a) = \frac{k}{(s-a)^2 + k^2}$

4b $u(t) := \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$ unit step function: turns things on and off! (at $t=0$)

Maple: Heaviside function

so $u(t-a) = \begin{cases} 1 & t \geq a \\ 0 & t < a \end{cases}$



$$\mathcal{L}\{u(t-a) f(t-a)\}(s) = \int_0^\infty \underbrace{e^{-st} u(t-a) f(t-a)}_{\text{integrand} = 0 \text{ until } t=a}\,dt = \int_a^\infty e^{-st} f(t-a)\,dt$$

subst $\tilde{t} = t-a$, $d\tilde{t} = dt$

$$= \int_{\tilde{t}=0}^\infty e^{-s(\tilde{t}+a)} f(\tilde{t})\,d\tilde{t}$$

$$= e^{-as}\int_0^\infty e^{-s\tilde{t}} f(\tilde{t})\,d\tilde{t}$$

$$= e^{-as} F(s) \; !$$

Exercise 1 Solve $\begin{cases} x'' + 6x' + 34x = 0 \\ x(0) = 3 \\ x'(0) = 1 \end{cases}$

with Laplace transform and completing the square.

Laplace:

$$s^2 X(s) - s\,x(0) - x'(0) + 6\,(s X(s) - x_0) + 34\,X(s) = 0$$

$$s^2 X(s) - 3s - 1 + 6\,(s\,X(s) - 3) + 34\,X(s) = 0$$

$$X(s)\,(s^2 + 6s + 34) = 3s + 19$$

$$X(s) = \frac{3s + 19}{s^2 + 6s + 34}$$

$$= \frac{3(s+3) + 10}{(s+3)^2 + 25}$$ ← then "complete the linear"

← complete the square

ans $x(t) = 3e^{-3t}\cos 5t + 2e^{-3t}\sin 5t$

$$= 3\,\frac{(s+3)}{(s+3)^2 + 25} + \frac{10}{(s+3)^2 + 25}$$

$f(t) = \cos 5t$

$F(s) = \frac{s}{s^2 + 25}$

$F(s+3) = \frac{s+3}{(s+3)^2 + 25}$

$\downarrow \mathcal{L}^{-1}$

$e^{-3t}\cos 5t$

$f(t) = \sin 5t$

$F(s) = \frac{5}{s^2 + 25}$

$F(s+3) = \frac{5}{(s+3)^2 + 25}$

$\downarrow \mathcal{L}^{-1}$

$e^{-3t}\sin 5t$

So, $x(t) = 3e^{-3t}\cos 5t + 2e^{-3t}\sin 5t$!

Exercise 2 Use partial fractions & table to find $f(t) = \mathcal{L}^{-1}\{F(s)\}(t)$, For

$$F(s) = \frac{s^2 + 3s + 1}{s^3 - 4s}$$

(Lindsay explains why this works! See Oct20.pdf page 4)

(Maple ans on next page)

Exercise 3 (Example 6 page 596, for worked out details)... a "different" way

This is your chance to review the ways to set up partial fracs!

Solve

$$\begin{cases} y^{(4)} + 2y'' + y = 4te^t \\ y(0)=0 \\ y'(0)=0 \\ y''(0)=0 \\ y'''(0)=0 \end{cases}$$

$$s^4Y(s) + 2s^2Y(s) + Y(s) = 4 \cdot \frac{1}{(s-1)^2}$$

$$(s^2+1)^2 Y(s) = \frac{4}{(s-1)^2} \quad \cdots$$

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> with(inttrans);   #integral transform package
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[*addtable, fourier, fouriercos, fouriersin, hankel, hilbert, invfourier, invhilbert, invlaplace, invmellin, laplace, mellin, savetable*]

#2

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> F:=s->(s^2+3*s+1)/(s^3-4*s);
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$$F := s \to \frac{s^2+3s+1}{s^3-4s}$$

```
> convert(F(s),parfrac,s);
```

$$\frac{11}{8(s-2)} - \frac{1}{4s} - \frac{1}{8(s+2)}$$

```
> invlaplace(F(s),s,t);
```

$$\frac{11}{8}e^{(2t)} - \frac{1}{4} - \frac{1}{8}e^{(-2t)}$$

#3

```
> Y:=s->4/((s-1)^2*(s^2+1)^2);
  convert(Y(s),parfrac,s);
  invlaplace(Y(s),s,t);
```

$$Y := s \to \frac{4}{(s-1)^2(s^2+1)^2}$$

$$\frac{2s}{(s^2+1)^2} + \frac{1}{(s-1)^2} - \frac{2}{s-1} + \frac{1+2s}{s^2+1}$$

$$(t-2)e^t + 2\cos(t) + (t+1)\sin(t)$$

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