Math 2250-1
Wednesday Nov. 5

HW for next Friday Nov 14
10.1 3 7 8, 13, 20, 21, 23, 28
10.2 3, 4 5, 6 14 19 20, 28 31
10.3 3 7, 8, 17, 20 27, 31
10.4 2, 3 9 10, 15, 16, 29, 36

Laplace transform magic!

Today we jump to chapter 10 (Laplace transform.) But don't worry, after chapter 10 we'll come back to 6-9.

But first, we need to talk about "variation of parameters" to find particular soltns to non-homogeneous DE's... §5.5. This was originally page 4 of Wed Oct 29:

<u>example</u> find y_P for $y''-4y=10e^{2x}$ } so on Oct 29 we guessed $y_P = Axe^{2x}$, and deduced $y_P=\frac{5}{2}xe^{2x}$

$y_H = c_1e^{2x}+c_2e^{-2x}$

<u>Variation of parameters</u> : to find y_p if you have a basis $\{y_1, y_2\}$ for y_H

$$\mathcal{L}(y) = y'' + p(x)y' + q(x)y$$

$$y_H = \text{span}\{y_1, y_2\} \quad \text{known.}$$

to solve

$$\mathcal{L}(y) = f$$

try $y_P = u_1y_1 + u_2y_2$ $\quad u_1, u_2 = $ fcns of x

$$\begin{aligned}
q(x)\,(\quad & y_P = u_1y_1+u_2y_2 \\
p(x)\,(\quad & y_P' = u_1y_1'+u_2y_2' + \underbrace{u_1'y_1+u_2'y_2}_{\text{set } \equiv 0} \\
1\,(\Rightarrow \quad & y_P'' = u_1y_1''+u_2y_2'' + \underbrace{u_1'y_1'+u_2'y_2'}_{\text{set } = f}
\end{aligned}$$

so $\mathcal{L}(y_P) = u_1\cancel{\mathcal{L}(y_1)}^{0} + u_2\cancel{\mathcal{L}(y_2)}^{0} + f$

$= f$

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f \end{bmatrix}$$

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \frac{1}{W}\begin{bmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{bmatrix}\begin{bmatrix} 0 \\ f \end{bmatrix}$$

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \frac{1}{W}\begin{bmatrix} -y_2f \\ y_1f \end{bmatrix}$$

integrate to get u_1 & u_2 then get y_P.

$y_1 = e^{2x}$
$y_2 = e^{-2x}$

$$W = \begin{vmatrix} e^{2x} & e^{-2x} \\ 2e^{2x} & -2e^{-2x} \end{vmatrix} = -4$$

$f = 10e^{2x}$

$$u_1' = -\tfrac{1}{4}\left(-e^{-2x}\,10e^{2x}\right) = \tfrac{5}{2}$$
$$u_2' = -\tfrac{1}{4}\left(e^{2x}\,10e^{2x}\right) = -\tfrac{5}{2}e^{4x}$$

take $u_1 = \frac{5}{2}x$
$u_2 = -\frac{5}{8}e^{4x}$

agrees with undetermined coef's soltn since $-\frac{5}{8}e^{2x}$ is a y_H

$$y_P = u_1y_1+u_2y_2 = \tfrac{5}{2}xe^{2x} - \tfrac{5}{8}e^{4x}e^{-2x} = \boxed{\tfrac{5}{2}xe^{2x} - \tfrac{5}{8}e^{2x}}$$

Chapter 10 : Laplace transform

a way of transforming IVP's for linear DE's, and linear systems of DE's, directly into solvable algebra problems.

let $f(t)$ be a function defined for $t \geq 0$ s.t. f grows at most exponentially fast ($|f(t)| \leq Ce^{Mt}$ some C, M)
Then the Laplace transform of f, $F(s) = \mathcal{L}\{f(t)\}(s)$
is defined for $s > M$ by

$$\underset{F(s)}{\underset{\parallel}{\mathcal{L}\{f(t)\}(s)}} := \int_0^\infty e^{-st} f(t)\, dt$$

examples

- $\mathcal{L}\{1\}(s) = \int_0^\infty e^{-st} 1 \, dt = \left.\frac{e^{-st}}{-s}\right]_0^\infty = \frac{1}{s} \qquad (s > 0)$
- $\mathcal{L}\{e^{at}\}(s) = \int_0^\infty e^{at} e^{-st} dt = \int_0^\infty e^{(a-s)t} dt = \left.\frac{1}{a-s} e^{(a-s)t}\right]_0^\infty = \frac{1}{s-a}$ $(s > a)$ (or if a is complex, $s >$ Real part of a)
- Laplace transform is linear!

$$\begin{aligned}
&\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\}(s) \\
&= \int_0^\infty e^{-st} (c_1 f_1(t) + c_2 f_2(t))\, dt \\
&= c_1 \int_0^\infty e^{-st} f_1(t)\, dt + c_2 \int_0^\infty e^{-st} f_2(t)\, dt \\
&= c_1 F_1(s) + c_2 F_2(s)
\end{aligned}$$

- $\mathcal{L}\{e^{ikt}\}(s) = \frac{1}{s-ik}$ see above!

$$\begin{aligned}
&\text{// Euler} && = \frac{s+ik}{s^2+k^2} \quad (\text{mult by } \tfrac{s+ik}{s+ik}) \\
&\mathcal{L}\{\cos kt + i \sin kt\}(s) && = \frac{s}{s^2+k^2} + i\frac{k}{s^2+k^2} \\
&\quad \parallel && \\
&\mathcal{L}\{\cos kt\}(s) + i\,\mathcal{L}\{\sin kt\}(s) &&
\end{aligned}$$

Thus (equate real parts; equate imag parts)

- $$\begin{cases} \mathcal{L}\{\cos kt\}(s) = \dfrac{s}{s^2+k^2} \\ \mathcal{L}\{\sin kt\}(s) = \dfrac{k}{s^2+k^2} \end{cases}$$

<u>Exercise</u>: What is $\mathcal{L}\{7 - 5e^{2t} + 4\cos 2t\}(s)$

<u>Laplace transform and DE's</u>:

$$\mathcal{L}\{f'(t)\}(s) = \int_0^\infty \underbrace{e^{-st}}_{v} \underbrace{f'(t)dt}_{du} = e^{-st}f(t)\Big]_{t=0}^{\infty} - \int_0^\infty -se^{-st}f(t)\,dt$$

$dv = -se^{-st}dt \quad u = f(t)$

$$= -f(0) + sF(s) \qquad s > M \leftarrow \text{exp growth rate of } f$$

so
$$\mathcal{L}\{f''(t)\}(s) = -f'(0) + s\mathcal{L}\{f'(t)\}(s)$$
$$= -f'(0) + s\left[-f(0) + sF(s)\right]$$
$$= s^2F(s) - sf(0) - f'(0)$$

<u>Exercise</u> Solve IVP with Laplace!! (i.e. without x_p, x_H, etc...)

$$* \begin{cases} x''(t) + 4x(t) = 10\cos 3t \\ x(0) = 2 \\ x'(0) = 1 \end{cases}$$

<u>Hard Theorem</u>: Laplace transform is invertible (and so the inverse is also linear)

$f(t)$	$F(s)$
1	$1/s$
e^{at}	$1/s-a$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
$\cos kt$	$\frac{s}{s^2+k^2}$
$\sin kt$	$\frac{k}{s^2+k^2}$
⋮	

we'll be filling this in! see e.g. book cover.

<u>Answer</u>: Write $\mathcal{L}\{x(t)\}(s) = X(s)$.
We know there is a unique sol'n to $*$.
get equality when Laplace both sides of the DE:

$$\mathcal{L}\{x''(t)\}(s) + 4\mathcal{L}\{x(t)\}(s) = 10\mathcal{L}\{\cos 3t\}(s)$$

$$s^2X(s) - s(2) - 1 + 4X(s) = 10\frac{s}{s^2+9}$$

$$X(s)(s^2+4) = \frac{10s}{s^2+9} + 1 + 2s$$

$$X(s) = \frac{10s}{(s^2+4)(s^2+9)} + \frac{1}{s^2+4} + \frac{2s}{s^2+4}$$

$$= 10s\left[\frac{1}{s^2+4} - \frac{1}{s^2+9}\right]\left[\frac{1}{5}\right] + \frac{1}{s^2+4} + \frac{2s}{s^2+4}$$

$$X(s) = -\frac{2s}{s^2+9} + \frac{1}{s^2+4} + \frac{4s}{s^2+4}$$

Table!! $\boxed{x(t) = -2\cos 3t + \tfrac{1}{2}\sin 2t + 4\cos 2t}$ Magic!!