Math 2250-1
Monday Nov. 3

Tomorrow is experiment day!
(among other things)

Finish §5.6 Forced harmonic oscillators

Comments on Friday notes: (Read along in class!)

- On page 3 we carefully solved the undamped forced oscillator eq'n when $\omega \neq \omega_0$

$$\begin{cases} x'' + \frac{k}{m} x = \frac{F_0}{m} \cos \omega t & \omega \neq \omega_0 = \sqrt{\frac{k}{m}} \\ x(0) = x_0 \\ x'(0) = v_0 \end{cases}$$

$$\boxed{x(t) = \frac{F_0}{m(\omega^2 - \omega_0^2)} (\cos \omega_0 t - \cos \omega t) + x_0 \cos \omega_0 t + \frac{v_0}{\omega_0} \sin \omega_0 t}$$

$\downarrow$ trig identities!

$$\boxed{x(t) = \frac{F_0}{m(\omega^2 - \omega_0^2)} \, 2 \sin(\underbrace{\bar{\omega}}_{\bar{\omega} = \frac{\omega + \omega_0}{2}} t) \sin(\underbrace{\delta}_{\delta = \frac{\omega - \omega_0}{2}} t) + x_0 \cos \omega_0 t + \frac{v_0}{\omega_0} \sin \omega_0 t}$$

second box explains beating

- On page 5 the notes derive the sol'n when ω does equal ω_0 (resonance)

$$\begin{cases} x'' + \frac{k}{m} x = \frac{F_0}{m} \cos \omega_0 t \\ x(0) = x_0 \\ x'(0) = v_0 \end{cases}$$

$$\boxed{x(t) = \frac{F_0}{2m\omega_0} t \sin(\omega_0 t) + x_0 \cos \omega_0 t + \frac{v_0}{\omega_0} \sin \omega_0 t}$$

Exercise 1 : We did not have time to do the derivation on page 5 Fri, of this sol'n. You might want to check it on your own.
But, there's a quick way to get this sol'n from the second "beating" box formula above by letting $\omega \to \omega_0$, so (for bounded t, $\sin \delta t \sim \delta t$. Try it!

Then check Example 3 on page 5 Friday.

Damped forced oscillations $(c>0)$

$$mx'' + cx' + kx = F_0 \cos\omega t$$

$$\begin{aligned}
k\,(\quad & x_p = A\cos\omega t + B\sin\omega t \\
+c\,(\quad & x_p' = -A\omega\sin\omega t + B\omega\cos\omega t \\
+m\,(\quad & x_p'' = -A\omega^2\cos\omega t - B\omega^2\sin\omega t
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}(x_p) = {} & \cos\omega t\,[kA + Bc\omega - mA\omega^2] \\
& + \sin\omega t\,[kB - Ac\omega - mB\omega^2]
\end{aligned} \quad = \cos\omega t\,(F_0)$$

$$\begin{bmatrix} k-m\omega^2 & c\omega \\ -c\omega & k-m\omega^2 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} F_0 \\ 0 \end{bmatrix}$$

so $$\begin{bmatrix} A \\ B \end{bmatrix} = \frac{1}{(k-m\omega^2)^2 + c^2\omega^2} \begin{bmatrix} k-m\omega^2 & -c\omega \\ c\omega & k-m\omega^2 \end{bmatrix} \begin{bmatrix} F_0 \\ 0 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} A \\ B \end{bmatrix} = \frac{1}{(k-m\omega^2)^2 + c^2\omega^2} \begin{bmatrix} F_0(k-m\omega^2) \\ F_0 c\omega \end{bmatrix}}$$

So $A\cos\omega t + B\sin\omega t$

$$= C\cos(\omega t - \alpha)$$

$$C = \frac{F_0}{(k-m\omega^2)^2 + c^2\omega^2}\sqrt{(k-m\omega^2)^2+c^2\omega^2} = \frac{F_0}{\sqrt{\underbrace{(k-m\omega^2)^2}_{\text{small if } \omega\approx\omega_0} + \underbrace{c^2\omega^2}_{\text{small if } c\approx 0}}}$$

$$x_H(t) = \begin{cases} c_1 e^{r_1 t} + c_2 e^{r_2 t} & r_1 < r_2 < 0 \quad \text{underdamped.} \\ e^{-pt}(c_1 + c_2 t) & \text{critically damped} \\ e^{-pt}(A\cos\omega_1 t + B\sin\omega_1 t) & \text{underdamped.} \end{cases}$$

So, full soltn

$$x(t) = \underbrace{x_p(t)}_{\text{periodic}} + \underbrace{x_H(t)}_{\substack{\text{decays exponentially do zero} \\ \text{regardless of choices of } c_1, c_2 \text{ (or } A, B)}}$$

So we call $x_p(t) = C\cos(\omega t - \alpha) = x_{sp}(t)$; steady periodic

$x_H(t) = x_{tr}(t)$; transient sol'n, reflects initial values *example*

$$\begin{cases} x'' + 2x' + 26x = 82\cos 4t \\ x(0) = 6 \\ x'(0) = 0 \end{cases}$$

could redo page 2 with numbers rather than letters (exam possibility), but we can also plug in:

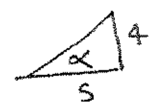
$$x_p(t) = A\cos 4t + B\sin 4t$$

$m = 1$
$\omega = 4$
$c = 2$
$k = 26$

$$\begin{bmatrix} A \\ B \end{bmatrix} = \frac{1}{100+64}\begin{bmatrix} 82\cdot 10 \\ 82\cdot 8 \end{bmatrix} = \begin{bmatrix} \frac{820}{164} \\ \frac{656}{164} \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

$k - m\omega^2 = 26 - 16 = 10$
$c\omega = 8$
$F_0 = 82$

$$x_p(t) = 5\cos 4t + 4\sin 4t = \sqrt{41}\cos(4t - \alpha)$$



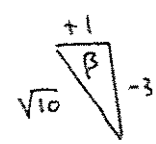
$\alpha = \arctan(.8)$.

$x_H(t)$:

$$r^2 + 2r + 26 = 0$$
$$(r+1)^2 + 25 = 0$$
$$(r+1+5i)(r+1-5i) = 0$$
$$r = -1 \pm 5i$$
$$x_H(t) = e^{-t}(A\cos 5t + B\sin 5t)$$

$$x(t) = 5\cos 4t + 4\sin 4t + e^{-t}(A\cos 5t + B\sin 5t)$$

$$\left.\begin{aligned} x(0) &= 6 = 5 + A \Rightarrow A = 1 \\ x'(0) &= 0 = 0 + 16 - A + 5B = 15 + 5B \Rightarrow B = -3 \end{aligned}\right\}$$



$$x(t) = \underbrace{\sqrt{41}\cos(4t - \alpha)}_{x_{sp}(t)} + \underbrace{e^{-t}\sqrt{10}\cos(5t - \beta)}_{x_{tr}(t)}$$

$\alpha = \arctan(.8)$ $\qquad$ $\beta = \arctan(-3)$

Practical resonance (see page 2)

$$x_{sp}(t) = \frac{F_0}{\sqrt{(k-m\omega^2)^2 + c^2\omega^2}} \cos(\omega t - \alpha)$$

if c is small
and $\omega^2 \approx \omega_0^2 = \frac{k}{m}$ $\Big\}$ denom tiny $\Rightarrow$ amplitude of x_{sp} huge (relative to F_0)

This is called <u>practical resonance</u>

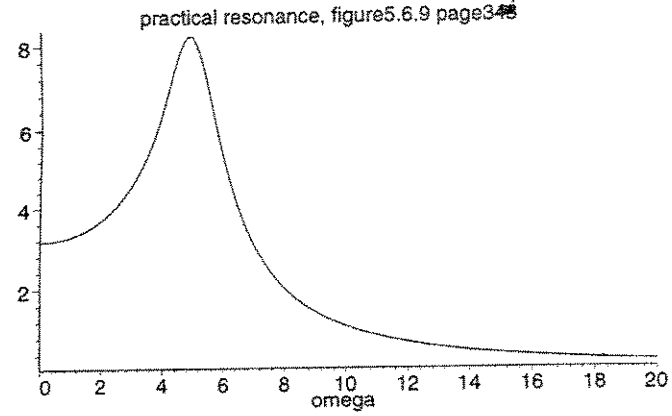
Vary ω in page 3 example

$$x'' + cx' + 26x = 82\cos\omega t$$

$$C = \frac{82}{\sqrt{(26-\omega^2)^2 + c^2\omega^2}} = C(\omega)$$

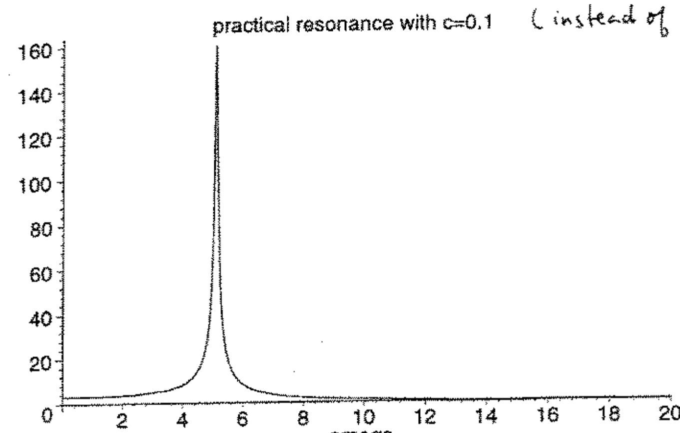
\> plot(82/sqrt((26-omega^2)^2+4*omega^2),omega=0..20,
color=black, title='practical resonance, figure5.6.9 page3~~48~~57');

practical resonance, figure5.6.9 page3~~48~~57



\> plot(82/sqrt((26-omega^2)^2+.01*omega^2),omega=0..20,
color=black, title='practical resonance with c=0.1');

practical resonance with c=0.1 (instead of 4)



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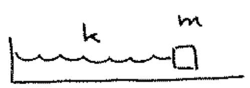
If you're an engineer concerned about resonance or practical resonance, you need to know how to figure out natural frequencies of mechanical systems.

The easiest way is often to use conservation of energy (essentially the antiderivative of Newton's law).

- We've done (via Newton)



mass - spring

Energy way:

$$KE + PE \equiv \text{const.}$$

$$\tfrac{1}{2} m (x'(t))^2 + \tfrac{1}{2} k x^2 \equiv \text{const}$$

$$\downarrow$$

$$W = \int_0^x F(x)dx = \int_0^x k s\, ds = \tfrac{1}{2} k s^2 \Big]_0^x = \tfrac{1}{2} k x^2$$

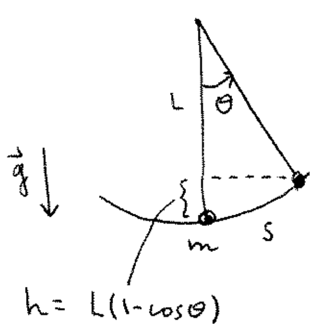
take D_t:

$$\tfrac{1}{2} m\, 2 x' x'' + \tfrac{1}{2} k\, 2 x x' \equiv 0$$

$$x' [\underbrace{m x'' + k x}_{\text{the DE!}}] \equiv 0$$

$$\boxed{\omega_0 = \sqrt{\tfrac{k}{m}}}$$

- We've done (via KE+PE)



$$KE + PE \equiv \text{const}$$

$$\tfrac{1}{2} m\, s'(t)^2 + mg h \equiv \text{const}$$

$$\tfrac{1}{2} m (L\theta')^2 + mg L (1-\cos\theta) \equiv \text{const}$$

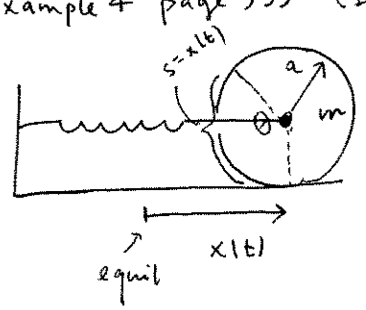
D_t: $m L^2 \theta' \theta'' + mg L (\sin\theta(t)) \theta'(t) \equiv 0$

$$L m \theta'(t) [L\theta'' + g \sin(\theta)] \equiv 0$$

$$\theta \text{ small} \rightarrow L\theta'' + g\theta \cong 0$$

$$\theta'' + \tfrac{g}{L}\theta = 0 \qquad \boxed{\omega_0 = \sqrt{\tfrac{g}{L}}}$$

- Example 4 page 353 (see also HW)



rolling wheel.

$$KE = \underbrace{\tfrac{1}{2} m (x')^2}_{\text{from motion of center of mass}} + \underbrace{\tfrac{1}{2} I \omega^2}_{\text{rotational KE}} \; ; \quad PE = \tfrac{1}{2} k x^2$$

$\omega = \frac{d\theta}{dt}$

I = moment of inertia $= \frac{ma^2}{2}$ for disks

$$x = a\theta$$

D_t: $x' = a\omega$; $\omega = \frac{x'}{a}$

$$KE + PE = \underbrace{\tfrac{1}{2} m (x')^2 + \tfrac{1}{2} \tfrac{1}{2} m a^2 \left(\tfrac{x'}{a}\right)^2}_{\frac{1}{4} m (x')^2} + \tfrac{1}{2} k x^2 = \tfrac{3}{4} m (x')^2 + \tfrac{1}{2} k x^2 \equiv \text{const}$$

D_t: $\tfrac{3}{2} m x' x'' + k x x' \equiv 0$

$$x' \left[\tfrac{3}{2} m x'' + k x\right] \equiv 0$$

$$\boxed{\omega_0 = \sqrt{\tfrac{2k}{3m}}}$$