Math 2250-1

Wednesday November 26

§7.4 Undamped spring systems

plan for the rest of the semester:

11/26 & 12/1	§7.4	12/8	9.2–9.3
12/2	§7.5	12/9	9.3–9.4
12/3	§9.1	12/10	9.4
12/5	§9.2	12/12	review entire course!!

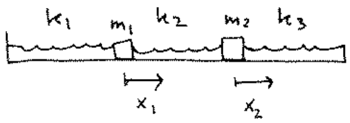
HW for Friday Dec. 5

7.4 2, 3, 8, 13, 14, 16, 17, 18

7.5 5, 6 (picture not required), 10, 15, 16, 19

& Maple (or alternative) exploration
in §7.4 → modeling a 7 story building
& how it might shake in an earthquake.
precise project will be posted.
The project due date will be Monday Dec. 8

Model this undamped spring system: (no damping)



$$m_1 x_1'' = k_2(x_2 - x_1) - k_1 x_1$$
$$m_2 x_2'' = -k_2(x_2 - x_1) - k_3 x_2$$

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix} = \begin{bmatrix} -k_1-k_2 & k_2 \\ k_2 & -k_2-k_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$M\vec{x}'' = K\vec{x}$$

is the general equation with n masses & up to $n+1$ springs in series

$$\Rightarrow \boxed{\vec{x}'' = A\vec{x}}$$ where $A = M^{-1}K$ is the "acceleration" matrix

in our example,

$$\begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix} = \begin{bmatrix} \frac{-k_1-k_2}{m_1} & \frac{k_2}{m_1} \\ \frac{k_2}{m_2} & \frac{-k_2-k_3}{m_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Strategy: We could convert this to a 1^{st} order system of 4 DE's, for x_1, x_1', x_2, x_2' (so soln space is 4 dim'l)

but (since there's no damping) here's a clever shortcut, motivated by the single eqtn case:

In general,
Look for sol'tns $\vec{x}(t) = \cos\omega t\, \vec{v}$ & $\sin\omega t\, \vec{v}$
and hope you get enough for a basis

$\vec{x} = \cos\omega t\, \vec{v}$

$\Rightarrow \vec{x}' = -\omega \sin\omega t\, \vec{v}$ & $A\vec{x} = \cos\omega t\, A\vec{v}$

$\Rightarrow \vec{x}'' = -\omega^2 \cos\omega t\, \vec{v}$ $\overset{?}{=}$

for $\boxed{\vec{x}'' = A\vec{x}}$ we see $\boxed{A\vec{v} = -\omega^2 \vec{v}}$ $\vec{v}$ is eigenvect of A, with eigenval $\lambda = -\omega^2$

- Hope A is diagonalizable.
  then each evect $\vec{v}$ yields 2 l.i. solns
  $\Rightarrow$ $2n$ lin ind solns $\Rightarrow$ basis!!

<u>Example</u> Let $k_1 = k_2 = k_3 = k$

$m_1 = m_2 = m$, so that the system on page 1 reduces to

$$\begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix} = \begin{bmatrix} -\frac{2k}{m} & \frac{k}{m} \\ \frac{k}{m} & -\frac{2k}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \frac{k}{m} \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Find the 4-dim'l solution space!

$\omega_1 = \sqrt{\frac{k}{m}}$ $\qquad$ $\omega_2 = \sqrt{\frac{3k}{m}}$

<u>answer</u>: $\vec{x}(t) = \underbrace{\left(c_1 \cos\omega_1 t + c_2 \sin\omega_1 t\right)}_{C_1 \cos(\omega_1 t - \alpha_1)} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \underbrace{\left(c_3 \cos\omega_2 t + c_4 \cos\omega_2 t\right)}_{C_2 \cos(\omega_2 t - \alpha_2)} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

slow, in-phase mode (masses oscillate in parallel)

faster, out of phase mode (masses oscillate in opposition)

Calculations for a 2 mass- 3 spring system

Math 2250-1,
November 26, 2008

The two mass, three spring system.

Data: Each ball mass is 50 grams. Each spring mass is 6 grams. (Remember, and this is a defect, our model assumes massless springs.) The springs are "identical", and an extra mass of 50 grams stretches the spring 18.0 centimeters from equilibrium. (We can recheck this.). Thus the spring constant is given by

```
> Digits:=5:
> solve(k*.18=.05*9.8,k);
```

$$2.7222$$

Let's time the two natural periods (which we discuss below):
(For the fast one, in my office, I got 50 cycles in about 25.14 seconds. (Hard to count this one!) For the slow one I got 20 cycles in about 18.03 seconds. What do we get in class?)

Here's the model:

```
> with(linalg):
> A:=matrix(2,2,[-2*k/m, k/m,k/m,-2*k/m]);
    #this should be the "A" matrix you get for
    #our two-mass, three-spring system.
```

$$A := \begin{bmatrix} -\frac{2k}{m} & \frac{k}{m} \\ \frac{k}{m} & -\frac{2k}{m} \end{bmatrix}$$

```
> eigenvects(A);
```

$$\left[-\frac{k}{m}, 1, \{[1, 1]\}\right], \left[-\frac{3k}{m}, 1, \{[-1, 1]\}\right]$$

Predict the two natural periods from the model:

(hopefully we just did this in class!)

ANSWER: If you do the model correctly and our data agrees with my office attempts, we will come up with natural periods of .49 and .85 seconds. I predict that the real natural periods are a little longer. What happened?

EXPLANATION: The springs actually have mass, equal to 6 grams each. This is not quite the same order of magnitude as the mass masses, and causes the actual experiment to run more slowly than our model predicts. In order to be more accurate the total energy of our model must account for the kinetic energy of the springs. You actually have the tools to model this more-complicated situation, using the ideas of total energy discussed in section 5.6, and a "little" Calculus. You can carry out this analysis, like I sketched for the single mass, single spring oscillator (nov4.pdf), assuming that the spring velocity at a point on the spring linearly interpolates the velocity of the wall and mass (or mass and mass) which bounds it. It turns out that this gives a non-diagonal M matrix, and ultimately an A-matrix the same eigenvectors, but different eigenvalues, namely

$$\lambda_1 = -\frac{k}{m + \frac{5}{6} m_s}$$

$$\lambda_2 = -3\frac{k}{m + \frac{1}{2} m_s}.$$

If you use these values, then you get period predictions

```
> m:=.05:
  ms:=.006;
  k:=2.722;
  Omega1:=sqrt(k/(m+(5/6)*ms));
  Omega2:=sqrt(3*k/(m+.5*ms));
  T1:=evalf(2*Pi/Omega1);
  T2:=evalf(2*Pi/Omega2);
```

$$ms := 0.006$$

$$k := 2.722$$

$$\Omega 1 := 7.0350$$

$$\Omega 2 := 12.412$$

$$T1 := 0.89316$$

$$T2 := 0.50621$$

of .89 and .51 seconds per cycle. Is that closer?

If we have time, consider forced oscillations,

$$M\vec{x}'' = K\vec{x} + \vec{F}(t)$$

(mult by M^{-1} on the left) $\quad * \quad \vec{x}'' = A\vec{x} + \vec{f}(t)$ $\qquad A = M^{-1}K$, $\vec{f} = M^{-1}\vec{F}$

suppose $\vec{f}(t) = \vec{F}_0 \cos\omega t$, i.e. you're forcing the masses at some angular frequency ω (the same for each mass being forced)

<u>assume</u> ω is not one of the natural frequencies for the unforced problem.

the general sol'n to $*$ will be

$$\vec{x}(t) = \vec{x}_p(t) + \vec{x}_H(t)$$

$\vec{x}_H(t)$: we figured out how to find this

$\vec{x}_p(t)$: try $\vec{x}_p(t) = \vec{c}\cos\omega t$ $\qquad$ (why don't we need another term too, $+\vec{d}\sin\omega t$?)

plug in to $*$!

$$\vec{c}(-\omega^2\cos\omega t) = A\vec{c}\cos\omega t + \vec{F}_0\cos\omega t$$

true for all time iff

$$\vec{c}(-\omega^2) = A\vec{c} + \vec{F}_0$$

$$-F_0 = (A + \omega^2 I)\vec{c}$$

$$\boxed{\vec{c} = -\underbrace{(A+\omega^2 I)^{-1}}_{}\vec{F}_0}$$

the inverse to this matrix will exist as long as $-\omega^2$ is <u>not</u> an eigenvalue of A, i.e. as long as ω is <u>not</u> a natural frequency.

... to be continued!