Math 2250-1
Tuesday November 29
§7.3, 7.2 : applications

We had this discussion yesterday, but it wasn't written in the notes:

If you're trying to find a basis of solutions to

$$\frac{d\vec{x}}{dt} = A\vec{x}$$

of the form $e^{\lambda t}\vec{v}$, how do you deal with complex eigenvalues and eigenvectors?

$\vec{x}_H$ for $\frac{d\vec{x}}{dt} = A\vec{x}$ when
- A is a real coeff matrix
- $\lambda = a + bi$ complex eval
  $\vec{v} = \vec{\alpha} + i\vec{\beta}$ complex evect.

Then

$\vec{z}(t) = e^{\lambda t}\vec{v}$ is a complex solution, since $\frac{d\vec{z}}{dt} = \lambda e^{\lambda t}\vec{v}$ and $A\vec{z} = Ae^{\lambda t}\vec{v} = e^{\lambda t}\lambda\vec{v}$ — still works, just as for real λ.

From this complex solution we extract 2 linearly independent solutions:

$$\vec{x}(t) = Re(e^{\lambda t}\vec{v})$$
$$\vec{y}(t) = Im(e^{\lambda t}\vec{v})$$

<u>check</u> $\vec{z}(t) = \vec{x}(t) + i\vec{y}(t)$

$\Rightarrow \vec{z}'(t) = \vec{x}'(t) + i\vec{y}'(t)$

$A\vec{z} = A\vec{x}(t) + i A\vec{y}(t)$

since $z' = Az$

$\Rightarrow \vec{x}' + i\vec{y}' = A\vec{x} + iA\vec{y}$

since real & imag parts must agree for complex #'s to be equal, deduce

$\vec{x}' = A\vec{x}$
& $\vec{y}' = A\vec{y}$

(one can check linear independence).

(You will get the same two sol'ns, up to sign, from $e^{(a+bi)t}\vec{v}$ and $e^{(a-bi)t}\bar{\vec{v}}$)

Modeling with first order systems of DE's

Math 2250-1
Tuesday November 25, 2008

Example 1: Tanks: This is example 2 from our text, page 418. We have a cascade of three tanks, see figure 7.3.2 page 418. The tank volumes are V1=20, V2=40, V3=50 (gallons). The water flow rate is r=10 g/min. The initial amounts of salt are x1(0)=15, x2(0)=0, x3(0)=0. (You actually had a 2-tank cascade way back in Chapter 1, which you solved in steps, without using systems of DEs.)

Show that the initial value problem for this system is

$$\begin{bmatrix} \frac{dx1}{dt} \\ \frac{dx2}{dt} \\ \frac{dx3}{dt} \end{bmatrix} = \begin{bmatrix} -.5 & 0 & 0 \\ .5 & -.25 & 0 \\ 0 & .25 & -.2 \end{bmatrix} \begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix}$$

$$\begin{bmatrix} \mathrm{x1}(0) \\ \mathrm{x2}(0) \\ \mathrm{x3}(0) \end{bmatrix} = \begin{bmatrix} 15 \\ 0 \\ 0 \end{bmatrix}$$

Solve this initial value problem, using

```
> with(linalg):
  Digits:=5:
> A:=matrix([[-.5, 0, 0], [.5, -.25, 0], [0, .25, -.2]]):
  eigenvectors(A);
```

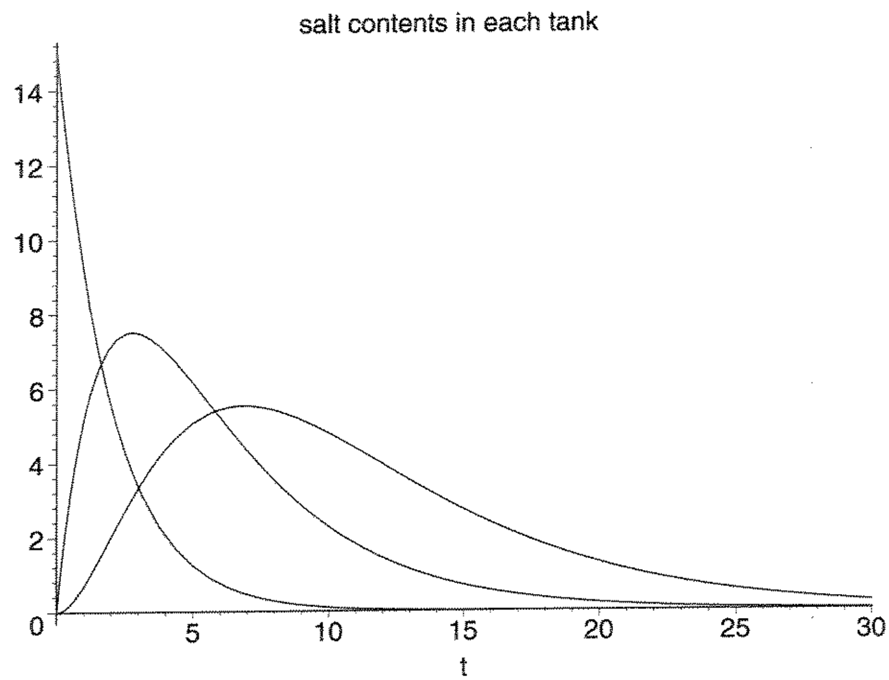
[-0.5, 1, {[1, -2.0000, 1.6667]}], [-0.25, 1, {[0, 1, -5.0000]}], [-0.2, 1, {[0, 0, 1]}]

You should get

$$\begin{bmatrix} x1(t) \\ x2(t) \\ x3(t) \end{bmatrix} = 5\,e^{(-0.5\,t)} \begin{bmatrix} 3 \\ -6 \\ 5 \end{bmatrix} + 30\,e^{(-0.25\,t)} \begin{bmatrix} 0 \\ 1 \\ -5 \end{bmatrix} + 125\,e^{(-0.2\,t)} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

We can plot the three solute amount vs. time to see what is going on:

```
> x1:=t->15*exp(-.5*t):
  x2:=t->-30*exp(-.5*t)+30*exp(-.25*t):
  x3:=t->25*exp(-.5*t)-150*exp(-.25*t)+125*exp(-.2*t):
> plot({x1(t),x2(t),x3(t)},t=0..30,color=black,
  title='salt contents in each tank');
```



Example 2: Glucose-insulin model (adapted from a discussion on page 339 of the text "Linear Algebra with Applications," by Otto Bretscher)

Let G(t) be the excess glucose concentration (mg of G per 100 ml of blood, say) in someone's blood, at time t hours. Excess means we are keeping track of the difference between current and equilibrium ("fasting") concentrations. Similarly, Let H(t) be the excess insulin concentration at time t. When blood levels of glucose rise, say as food is digested, the pancreas reacts by secreting insulin in order to utilize the glucose. Researchers have developed mathematical models for the glucose regulatory system. Here is a simplified (linearized) version of one such model, with particular representative matrix coefficients. It would be meant to apply between meals, when no additional glucose is being added to the system:

$$\begin{bmatrix} \dfrac{dG}{dt} \\ \dfrac{dH}{dt} \end{bmatrix} = \begin{bmatrix} -.1 & -.4 \\ .1 & -.1 \end{bmatrix} \begin{bmatrix} G \\ H \end{bmatrix}$$

Explain (understand) the signs of the matrix coefficients:

Now let's solve the initial value problem, say right after a big meal, when

$$\begin{bmatrix} G(0) \\ H(0) \end{bmatrix} = \begin{bmatrix} 100 \\ 0 \end{bmatrix}$$

```
> with(plots):
> A:=matrix(2,2,[-.1,-.4,.1,-.1]);
```

$$A := \begin{bmatrix} -0.1 & -0.4 \\ 0.1 & -0.1 \end{bmatrix}$$

```
> eigenvectors(A);
```

$$[-0.1 + 0.20000\, I,\ 1,\ \{[-2.0000 + 0.\, I,\ 0. + 1.\, I]\}],\ [-0.1 - 0.20000\, I,\ 1,\ \{[-2.0000 - 0.\, I,\ 0. - 1.\, I]\}]$$

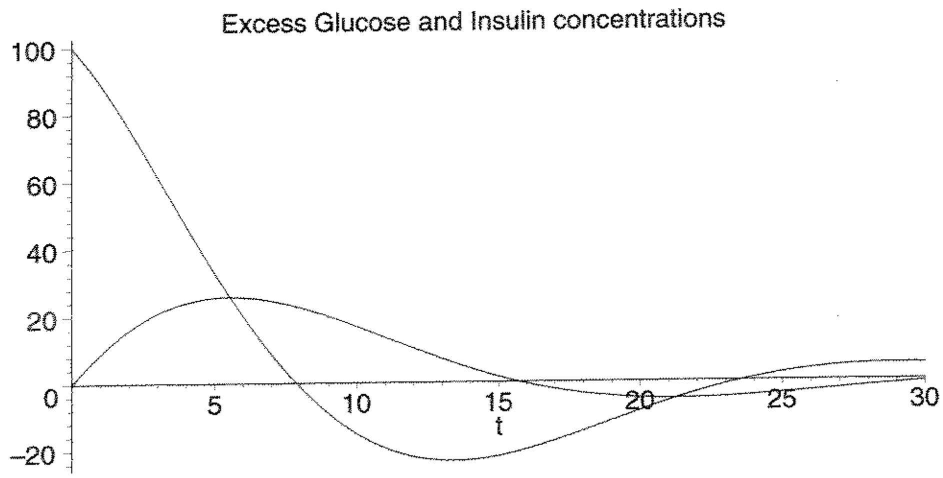
Can you get the same eigenvalues and eigenvectors? Notice that Maple writes a capital I for the square root of -1, i. Extract a basis for the solution space to his homogeneous system of differential equations from the eigenvector information above:

Solve the initial value problem.

Here are some pictures to help understand what the model is predicting:

(1) Plots of glucose vs. insulin, at time t hours later:

```
> G:=t->100*exp(-.1*t)*cos(.2*t):
  H:=t->50*exp(-.1*t)*sin(.2*t):
> plot({G(t),H(t)},t=0..30,color=black,title=
  `Excess Glucose and Insulin concentrations`);
```



2) A phase portrait of the glucose-insulin system:

```
> pict1:=fieldplot([-.1*G-.4*H,.1*G-.1*H],G=-40..100,H=-15..40):
  soltn:=plot([G(t),H(t),t=0..30],color=black):
  display({pict1,soltn},title=`Glucose vs Insulin
  phase portrait`);
```