Math 2250-1

Monday November 24

§7.2-7.3

We've been discussing first order systems of differential equations, and the natural initial value problem

$$\begin{cases} \frac{d\vec{x}}{dt} = \vec{f}(t, \vec{x}) \\ \vec{x}(t_0) = \vec{x}_0 \end{cases}$$

(and that any system of higher order differential equations can be studied in terms a related system of 1^{st} order DE's.)

- A good place to start again is <u>page 4 Friday</u>, the undamped, unforced harmonic oscillator re-interpreted as a system of 1^{st} order DE's.
- Then:

<u>General result</u>

<u>IVP</u> $\begin{cases} \frac{d\vec{x}}{dt} = \vec{f}(t, \vec{x}) \\ \vec{x}(t_0) = \vec{x}_0 \end{cases}$

If $\vec{f}$ is differentiable in its variables the there exists a unique sol'n to IVP, defined on some interval $(t_0 - \delta, t_0 + \delta)$

<u>Linear systems of DE's result</u>

<u>IVP</u> $\begin{cases} \frac{d\vec{x}}{dt} = P(t)\vec{x} + \vec{f}(t) \\ \vec{x}(t_0) = \vec{x}_0 \end{cases}$

- $P(t)$ is an $n \times n$ matrix fcn of t
- system is homogeneous if $\vec{f}(t) \equiv 0$ (else inhomogeneous)

If $P(t)$ and $\vec{f}(t)$ are continuous on an interval containing t_0, then there exists a unique sol'n to IVP, defined on the entire interval.

The intuition for these results is developed in the Maple handout, and in Wed's notes, interpreting $\frac{d\vec{x}}{dt}$ as the "velocity" vector of position fcn $\vec{x}(t)$

Systems of linear 1^{st} order DE's theory

(this should feel familiar)

1 Homogeneous system

$$* \quad \frac{d\vec{x}}{dt} = P(t)\vec{x}$$

- solution space is a subspace of $\vec{\tilde{\mathcal{F}}}$: vector-valued fcns of t

  Let $\vec{x}(t)$ and $\vec{z}(t)$ solve $*$

  Then

$$\frac{d}{dt}\left(c_1\vec{x}(t) + c_2\vec{z}(t)\right) = c_1\frac{d\vec{x}}{dt} + c_2\frac{d\vec{z}}{dt}$$

$$= c_1\left(P(t)\vec{x}\right) + c_2\left(P(t)\vec{z}\right) \qquad \text{because } \vec{x}(t),\ \vec{z}(t) \text{ are sol'ns}$$

$$= P(t)\left(c_1\vec{x}(t) + c_2\vec{z}(t)\right) \qquad \text{Matrix properties}$$

  so linear combos of sol'ns are sol'ns $\rightarrow$ this condition (closure under + & scalar mult) is the subspace condition.

- the solution subspace to $*$ is n dimensional $\left(\text{ if } \vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}\right)$

  Because, if

$$\vec{x}_j(t) \text{ solves } \begin{cases} \frac{d\vec{x}}{dt} = P(t)\vec{x} \\ \vec{x}_j(t_0) = \vec{e}_j = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow \text{entry } j \end{cases}$$

  then $\{\vec{x}_1(t), \vec{x}_2(t), \ldots \vec{x}_n(t)\}$ are a basis :

  <u>span</u> : The sol'n to $\begin{cases} \frac{d\vec{x}}{dt} = P(t)\vec{x} \\ \vec{x}(t_0) = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \end{cases}$

  is $\vec{x}(t) = b_1\vec{x}_1(t) + b_2\vec{x}_2(t) + \ldots + b_n\vec{x}_n(t)$

  <u>independent</u> : if $c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + \ldots + c_n\vec{x}_n(t) \equiv \vec{0}$

  then at t_0 : $c_1\vec{e}_1 + c_2\vec{e}_2 + \ldots + c_n\vec{e}_n = \vec{0}$

$$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

2 Inhomogeneous 1^{st}-order systems of DE's.

$$\frac{d\vec{x}}{dt} = P(t)\vec{x} + \vec{f}(t)$$

The general solution is

$$\vec{x}(t) = \vec{x}_p(t) + \vec{x}_H(t)$$

where $\vec{x}_p(t)$ is a particular sol'n and $\vec{x}_H(t)$ is the general homogeneous sol'n

<u>proof</u> $\mathcal{L}(\vec{x}) := \frac{d\vec{x}}{dt} - P(t)\vec{x}$ is a linear operator so the proof is virtually identical to the discussion for nonhomogeneous linear differential equations of order n.

Example

$$\frac{d\vec{x}}{dt} = \underbrace{\begin{bmatrix} 3 & -2 & 0 \\ -1 & 3 & -2 \\ 0 & -1 & 3 \end{bmatrix}}_{P(t)} \vec{x} + \underbrace{\begin{bmatrix} 5 \\ -15 \\ 10 \end{bmatrix}}_{\vec{f}(t)}$$

$\vec{x}_H(t)$:

```
> with(linalg):
  A:=matrix(3,3,[3,-2,0,-1,3,-2,0,-1,3]);
  eigenvectors(A);
```

$$A := \begin{bmatrix} 3 & -2 & 0 \\ -1 & 3 & -2 \\ 0 & -1 & 3 \end{bmatrix}$$

$$[5, 1, \{[2, -2, 1]\}], [3, 1, \{[-2, 0, 1]\}], [1, 1, \{[2, 2, 1]\}]$$

eigenvalue

algebraic multiplicity, i.e. $(\lambda-5)^1$ is the factor in characteristic poly

eigenbasis

Maple tells us

$$\vec{x}_H(t) = c_1 e^{5t}\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} + c_2 e^{3t}\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + c_3 e^{t}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$

$\leftarrow$ Note "Wronskian" $W(t) = \left| e^{5t}\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, e^{3t}\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, e^{t}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \right|$

(if our system came by converting a DE, this is the old Wronskian!)

$$= e^{9t}\begin{vmatrix} 2 & -2 & 2 \\ -2 & 0 & 2 \\ 1 & 1 & 1 \end{vmatrix} = e^{9t}(-32) \neq 0$$

so our 3 vector-valued sol's are linearly independent

So we have all sol's, i.e.

$$\left\{ e^{5t}\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, e^{3t}\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, e^{t}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \right\}$$

are a basis (they're independent and span because we can solve any IVP with them.)

$\vec{x}_p(t)$: Try $\vec{x}_p(t) = \vec{c}$ (a constant vector)

then $\frac{d\vec{x}_p}{dt} = \frac{d}{dt}\vec{c} = 0 \overset{?}{=} A\vec{c} + \begin{bmatrix} 5 \\ -15 \\ 10 \end{bmatrix}$

$$\vec{c} = -A^{-1}\begin{bmatrix} 5 \\ -15 \\ 10 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}$$

```
> v:=vector([5,-15,10]);
  x:=evalm(-inverse(A)&*v);
```

$v := [5, -15, 10]$

$x := [1, 4, -2]$

```
> 0=evalm(A&*x+v);   #check DE
```

$0 = [0, 0, 0]$

Full solution

$$\vec{x}(t) = \vec{x}_p + \vec{x}_H$$

$$\boxed{\vec{x}(t) = \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix} + c_1 e^{5t}\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} + c_2 e^{3t}\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + c_3 e^{t}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}}$$