Math 2250-1
Friday November 21
7.1, with hints of 7.2, 7.3

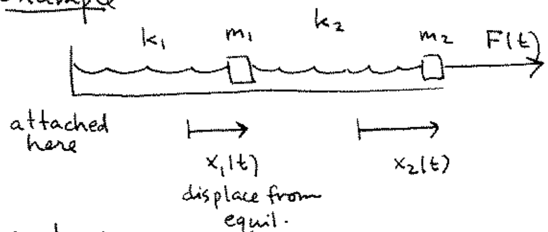
HW for Monday Dec 1
(next week is Thanksgiving week)

7.1 1, 3, 5, 8, 11, 18, 21, 24, 26
7.2 5, 9, 10, 13, 14, 16, 21, 23, 25
7.3 3, 4, 6, 13, 14, 25, 30, 32, 37

To motivate our study of eigenvalues/eigenvectors, we studied a tank system (nov 12) of DE's. So, it makes sense to ask:

- What kind of DE's do we get for an interconnected mass-spring system?

Example



Newton:

$$m_1 x_1''(t) = k_2 \underbrace{(x_2 - x_1)}_{\text{2nd spring is stretched this much}} - k_1 x_1$$

$$m_2 x_2''(t) = -k_2 (x_2 - x_1) + F(t)$$

e.g. $m_1 = 2$, $m_2 = 1$, $k_1 = 4$, $k_2 = 2$, $f(t) = 40 \sin 3t$ $\Rightarrow$

$$\begin{cases} 2x_1'' = 2(x_2 - x_1) - 4x_1 = -6x_1 + 2x_2 \\ x_2'' = -2(x_2 - x_1) + 40 \sin 3t = 2x_1 - 2x_2 + 40 \sin 3t \end{cases}$$

or $\begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 40 \sin 3t \end{bmatrix}$

- What's a natural IVP for this 2nd order system of 2 linear DE's? (We won't solve it today – see § 7.4)

<u>note</u>: Any system of DE's can be converted into an equivalent system of 1st order DE's, by introducing extra unknown functions.

(1) $$\boxed{\begin{aligned} x_1'' &= -3x_1 + x_2 \\ x_2'' &= 2x_1 - 2x_2 + 40 \sin 3t \end{aligned}}$$

Let $y_1 = x_1'$
$y_2 = x_2'$

(2) $$\boxed{\begin{aligned} x_1' &= y_1 \\ y_1' &= -3x_1 + x_2 \\ x_2' &= y_2 \\ y_2' &= 2x_1 - 2x_2 + 40 \sin 3t \end{aligned}}$$

solutions of (1) yield solutions of (2) <u>and</u> vise versa.

This correspondence is sometimes useful.

<u>Example</u> : Study the correspondence between n^{th} order eqtns or systems and 1^{st} order ones : (Every system of arbitrary order DE's can be reduced to an equivalent first order system. The correspondence doesn't always go in the reverse direction)

$x = x(t)$

<u>2^{nd} order linear homog DE</u>

$$x'' - x' - 2x = 0$$

$x = e^{rt}$

$p(r) = r^2 - r - 2 = 0$

$(r-2)(r+1) = 0$

$$\boxed{x_H(t) = c_1 e^{2t} + c_2 e^{-t}}$$

so we could immediately solve the 1^{st} order system :

$x_H(t) = c_1 e^{2t} + c_2 e^{-t}$

$x_H'(t) = 2c_1 e^{2t} - c_2 e^{-t}$

so $\begin{bmatrix} x \\ y \end{bmatrix} = c_1 e^{2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Also, see work at right!

<u>IVP</u> $\begin{cases} x'' - x' - 2x = 0 \\ x(0) = 2 \\ x'(0) = 1 \end{cases}$

$c_1 + c_2 = 1$
$2c_1 - c_2 = 1$

$$\boxed{x(t) = e^{2t} + e^{-t}}$$

<u>1^{st} order system of 2 DE's</u>

$y = x'$

$x' = y$
$y' = y + 2x$ $\quad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$.

try $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = e^{\lambda t} \vec{v}$

for $\vec{x}'(t) = A\vec{x}$

$\lambda e^{\lambda t} \vec{v} = A e^{\lambda t} \vec{v}$

$\lambda \vec{v} = A\vec{v}$ $\quad$ $\vec{v}$ eigenvect of A

$0 = (A - \lambda I)\vec{v}$ $\quad$ eigenval λ

$\det \begin{bmatrix} -\lambda & 1 \\ 2 & 1-\lambda \end{bmatrix} = \lambda^2 - \lambda - 2 = p(\lambda)$

does this "characteristic polynomial" look familiar?

$= (\lambda - 2)(\lambda + 1)$

$\lambda = 2$: $\left[\begin{array}{cc|c} -2 & 1 & 0 \\ 2 & -1 & 0 \end{array}\right]$, $\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\lambda = -1$: $\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 2 & 2 & 0 \end{array}\right]$, $\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

eigenbasis

$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ ✓

IVP $\begin{cases} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \end{cases}$

$\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

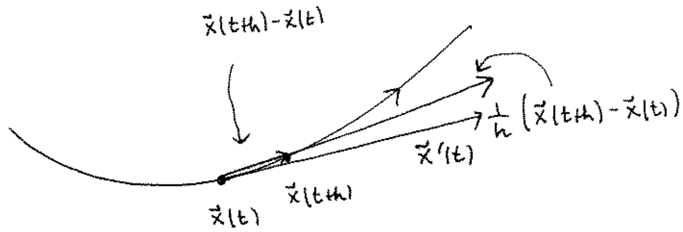
$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = e^{2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Geometric interpretation of 1^{st} order system of DE's:

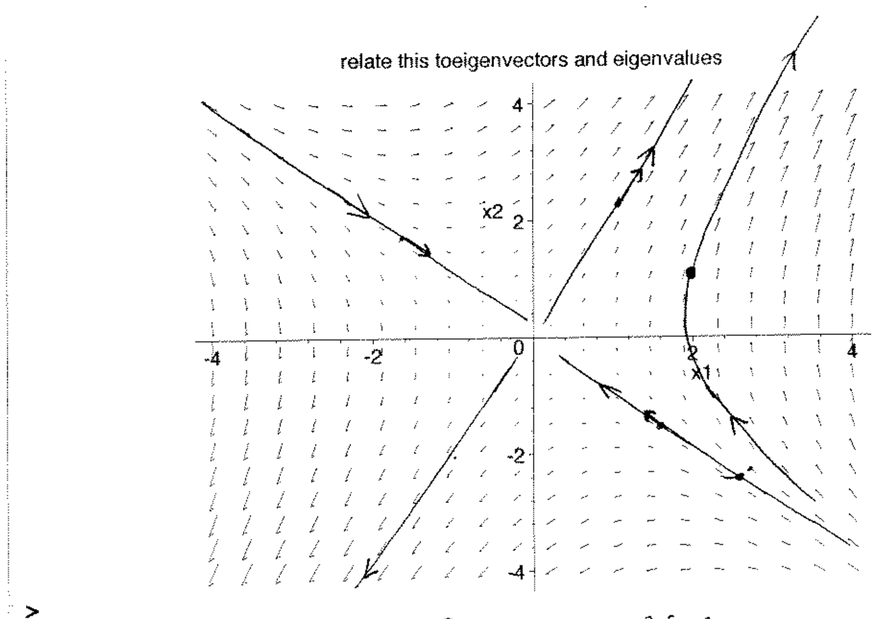
IVP $\begin{cases} \dfrac{d\vec{x}}{dt} = \vec{F}(t,\vec{x}) \\ \vec{x}(t_0) = \vec{x}_0 \end{cases}$

- Recall, if $\vec{x}(t)$ is parametric curve in space (think partical position at time t) then $\vec{x}'(t)$ is the tangent (or velocity) vector:



So the IVP says you know where you start ($\vec{x}(t_0)=\vec{x}_0$), and you know your velocity vector (depending on time & location) $\longrightarrow$ so you expect a unique sol'n

<u>page 2 example continued</u> :



$$\begin{cases} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \end{cases}$$

The Harmonic oscillator
(unforced, undamped).

$$x'' + x = 0$$

$x' = y$

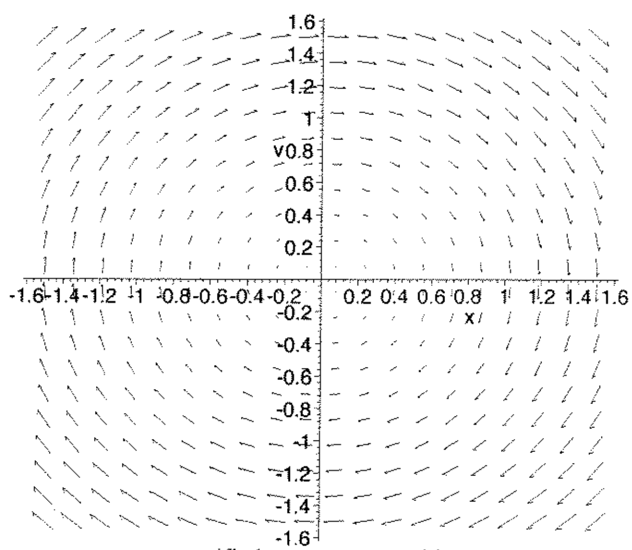
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} y \\ -x \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$x(t) = C\cos(t-\alpha)$$

so $\begin{bmatrix} x \\ x' \end{bmatrix} = C \begin{bmatrix} \cos(t-\alpha) \\ -\sin(t-\alpha) \end{bmatrix}$.

clockwise circles!

phase portrait for the harmonic oscillator



$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} y \\ -x \end{bmatrix}$$

what if you try solving this 1^{st} order system of DE's with the eigenvalue - eigenvector method?