Math 2250-1

Tuesday November 17 6.2-6.3

If $A_{n\times n}$, what is an

- eigenvalue?
- eigenvector?
- eigenspace?

When do we call A diagonalizable?

Examples from last week

$A_1 = \begin{bmatrix} -.1 & .2 \\ .1 & -.2 \end{bmatrix}$ $|A_1 - \lambda I| = \lambda(\lambda + .3)$

$\lambda = 0$ $\lambda = -.3$

$\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ is a basis of $\mathbb{R}^2$ - A_1 is dia

Exercise 1 Let P be the matrix which has the eigenvectors as columns

$P = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$

- Find $P^{-1} =$

- Show $P^{-1}AP = \begin{bmatrix} 0 & 0 \\ 0 & -.3 \end{bmatrix}$ is diagonal (and the diagonal entries are the eigenvalues of the corresponding eigenvectors

- Since $P^{-1}AP = D$

$P(P^{-1}AP)P^{-1} = PDP^{-1}$

$A = PDP^{-1}$

Compute A^{10} by hand

for $A_2 = \begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix}$ $\quad |A_2 - \lambda I| = -(\lambda-2)^2(\lambda-3)$ (see last week's notes)

$\lambda = 2$ eigenspace basis $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \right\}$; $\lambda = 3$ eigenbasis $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^3$

<u>Exercise 2</u> Let $P = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}$, $P^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ -1 & 1 & 0 \\ 2 & -2 & 1 \end{bmatrix}$.

- Compute $P^{-1}A_2P = \begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix}$ (a certain diagonal matrix!)

- Express A^{100} as a product which only requires two matrix multiplications to complete.

<u>Theorem</u> If $\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n\}$ is a basis of $\mathbb{R}^n$ (or $\mathbb{C}^n$) consisting of eigenvectors of A, and if P is the matrix $\left[\vec{v}_1 \middle| \vec{v}_2 \middle| \cdots \vec{v}_n \right]$.

then $P^{-1}AP = D = \begin{bmatrix} \lambda_1 & & O \\ & \lambda_2 & \\ O & & \ddots \; \lambda_n \end{bmatrix}$ is a diagonal matrix, with the eigenvalues down the diagonal; $A\vec{v}_j = \lambda_j \vec{v}_j$

So also $A = PDP^{-1}$
$A^k = PD^kP^{-1}$.

(It turns out that for any matrix A, the dimension of the λ_j-eigenspace is <u>at most</u> the power that $(\lambda - \lambda_j)$ appears in the characteristic polynomial. If these numbers are actually equal for all the λ_j, then this is equivalent to A being diagonalizable, and your $\mathbb{R}^n$ (or $\mathbb{C}^n$) basis is just the union of the different eigenbases)

Example (this is Example 2. p 382. It will lead to an explanation of the Google algorithm tomorrow).

A $\underbrace{\text{metropolitan area}}_{\text{city \& suburbs.}}$ has a constant population of 1 million people

Let C_k, S_k be # of city & suburban dwellers in year k.

Suppose each year 15% of city dwellers move to suburbs
10% of suburb dwellers move to city:

so

$$C_{k+1} = .85 C_k + .1 S_k$$
$$S_{k+1} = .15 C_k + .9 S_k$$

$$\begin{bmatrix} C_{k+1} \\ S_{k+1} \end{bmatrix} = \begin{bmatrix} .85 & .1 \\ .15 & .9 \end{bmatrix} \underbrace{\begin{bmatrix} C_k \\ S_k \end{bmatrix}}_{[\ ]\begin{bmatrix} C_{k-1} \\ S_{k-1} \end{bmatrix}}$$
$$= \dots$$
$$\begin{bmatrix} C_{k+1} \\ S_{k+1} \end{bmatrix} = \begin{bmatrix} .85 & .1 \\ .15 & .9 \end{bmatrix}^{k+1} \begin{bmatrix} C_0 \\ S_0 \end{bmatrix}$$

this is an example of a discrete dynamical system, with a constant transition matrix

Describe what happens long term!

$$A = \begin{bmatrix} .85 & .1 \\ .15 & .9 \end{bmatrix} \qquad |A-\lambda I| = \begin{vmatrix} .85-\lambda & .1 \\ .15 & .9-\lambda \end{vmatrix} = (.85-\lambda)(.9-\lambda) - .015 = \lambda^2 - 1.75\lambda - .75$$
$$= (\lambda-1)(\lambda-.75)$$

$\lambda = 1$

$$\left[\begin{array}{cc|c} -.15 & .1 & 0 \\ .15 & -.1 & 0 \\ \hline -3/2 & 1 & 0 \\ 0 & 0 & 0 \\ \hline -3 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad \vec{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$\lambda = .75$

$$\left[\begin{array}{cc|c} .1 & .1 & 0 \\ .15 & .15 & 0 \\ \hline 1 & 1 & 0 \\ 0 & 0 & \\ \hline \end{array}\right] \qquad \vec{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

$$P = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix} \qquad P^{-1} = \frac{1}{5}\begin{bmatrix} 1 & 1 \\ -3 & 2 \end{bmatrix}$$

$$P^{-1}AP = D = \begin{bmatrix} 1 & 0 \\ 0 & .75 \end{bmatrix}$$

$$A = PDP^{-1}$$
$$A^k = PD^kP^{-1}$$
$$= P\begin{bmatrix} 1 & 0 \\ 0 & .75 \end{bmatrix}P^{-1}$$

$$\begin{bmatrix} C_k \\ S_k \end{bmatrix} = A^k \begin{bmatrix} C_0 \\ S_0 \end{bmatrix}$$
$$= P\begin{bmatrix} 1 & 0 \\ 0 & (.75)^k \end{bmatrix}P^{-1}\begin{bmatrix} C_0 \\ S_0 \end{bmatrix}$$
$$= \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & .75^k \end{bmatrix}\frac{1}{5}\begin{bmatrix} 1 & 1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} C_0 \\ S_0 \end{bmatrix}$$

k large $\approx \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\frac{1}{5}\begin{bmatrix} 1 & 1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} C_0 \\ S_0 \end{bmatrix}$

$$= \frac{1}{5}\begin{bmatrix} 2 & 0 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} C_0 \\ S_0 \end{bmatrix}$$
$$= \frac{1}{5}\begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} C_0 \\ S_0 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{2}{5}(C_0+S_0) \\ \frac{3}{5}(C_0+S_0) \end{bmatrix} \begin{matrix} \longleftarrow 2/5 \text{ eventually in the city} \\ \longleftarrow 3/5 \text{ eventually is the suburbs!} \end{matrix}$$