Math 2250-1
Friday November 14

Exam Monday!
Review sheet posted & at end of today's notes.
Practice exam will be (or is) posted.
Review session Saturday 11:00-12:30 JWB 335

HW for next Friday is posted & in Wednesday notes.

§6.1-6.2
On Wednesday we saw how modeling an input/output (e.g. tank) system can lead to a system of differential equations

$$* \qquad \vec{x}\,'(t) = A\vec{x} \qquad \text{(where } A \text{ is an } n\times n \text{ matrix and)}$$

where $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$ and $x_j(t)$ is the amount of solute in tank j.

The solution space to $*$ is a subspace, and trying to find a basis of solutions led us to an interesting algebra problem:

$\vec{v}$ is an eigenvector for A, with eigenvalue λ means $A\vec{v} = \lambda\vec{v}$.

- Can we find a basis for $\mathbb{R}^n$ (or, actually for $\mathbb{C}^n$) made out of eigenvectors of A? (If we can, we'll be able to find a basis of soltns to $*$, $\{e^{\lambda_1 t}\vec{v}_1, e^{\lambda_2 t}\vec{v}_2, \ldots e^{\lambda_n t}\vec{v}_n\}$ so that the general soln to $*$ is $\vec{x}_H(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + \ldots + c_n e^{\lambda_n t}\vec{v}_n$!)

Exercise 1 Work the example on page 3 of Wednesday notes → try to find an $\mathbb{R}^3$ basis made of eigenvectors of

$$A = \begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix}$$

(details in Wed. notes).
Recall the algorithm!

Definition We call A diagonalizable if there exists a basis for $\mathbb{R}^n$ (or later, $\mathbb{C}^n$), made out of eigenvectors of A

So, $A = \begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix}$ and $A = \begin{bmatrix} -.1 & .2 \\ .1 & -.2 \end{bmatrix}$ are diagonalizable

Exercise 2: Show that $B = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ is not diagonalizable

ans: $|B - \lambda I| = \begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = -(\lambda-2)^2(\lambda-3)$

$\lambda = 2$:

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$v_3 = 0$
$v_2 = 0$
$v_1 = t$ $\quad \vec{v} = t\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

eigenspace is 1-dim'l!
$= \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}$

only have two independent eigenvectors!

$\lambda = 3$

$$\left[\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\downarrow$ rref

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$v_3 = t$
$v_2 = 0$
$v_1 = 0$

$\text{span}\left\{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$.

Here's why we call a matrix diagonalizable if we can make an $\mathbb{R}^n$ (or $\mathbb{C}^n$) basis from its eigenvectors:

Theorem A is diagonalizable if and only if there is a diagonal matrix D ~~so that~~ and an invertible matrix P so that

$$D = P^{-1}AP \qquad (\text{or, equivalently, } PDP^{-1} = A)$$

In this case the columns of P are a basis of eigenvectors of A!

proof Both equations are equivalent to

$$PD = AP.$$

Write $P = \left[\vec{v}_1 \middle| \vec{v}_2 \middle| \cdots \vec{v}_n\right]$ in terms of its col'n

and $D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$

$$\left[\vec{v}_1 \middle| \vec{v}_2 \middle| \cdots \vec{v}_n\right] \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} = A\left[\vec{v}_1 \middle| \vec{v}_2 \middle| \cdots \vec{v}_n\right]$$

just says $\left[\lambda_1\vec{v}_1 \middle| \lambda_2\vec{v}_2 \middle| \cdots \lambda_n\vec{v}_n\right] = A\left[\vec{v}_1 \middle| \vec{v}_2 \middle| \cdots \vec{v}_n\right]$, i.e. $A\vec{v}_j = \lambda\vec{v}_j$. ∎

Notice, if $A = PDP^{-1}$

then $A^2 = PD\underbrace{P^{-1}P}_{I}DP^{-1} = PD^2P^{-1}$

$A^3 = A^2 A = PD^2P^{-1}PDP^{-1} = PD^3P^{-1}$

$\vdots$

$A^k = PD^kP^{-1}$ (and D^k is just the diagonal matrix of k^{th} powers of the original diagonal matrix)

Exercise 3 Check $A = PDP^{-1}$ (or equivalent) for fill in!

a) $A = \begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix}$, $P = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}$, $D = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$

$\lambda = 2$ eigenbasis $\lambda = 3$ eigenbasis

b) $A = \begin{bmatrix} -.1 & .2 \\ .1 & -.2 \end{bmatrix}$, $P = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$ $D = \begin{bmatrix} 0 & 0 \\ 0 & -.3 \end{bmatrix}$

$\lambda = 0$ eigenbasis $\lambda = -.3$ eigenbasis

Exercise 4 Compute A^{10} for one of the matrices in 3, using $A = PDP^{-1}$

Are there times you know a matrix will be diagonalizable (even before you look for all the eigenvectors?)

<u>Theorem 1</u> Let $\vec{v}_1, \vec{v}_2, \dots \vec{v}_k$ be eigenvectors (non-zero), of A, with different eigenvalues, i.e. $A\vec{v}_j = \lambda_j \vec{v}_j$ with $\lambda_j = \lambda_i$ only when $i=j$.

Then $\vec{v}_1, \vec{v}_2, \dots \vec{v}_k$ are linearly independent.

<u>proof</u>: By induction on k: if $k=1$, we have one non-zero eigenvector $\vec{v}_1$, any non-zero vector is independent, since $c_1\vec{v}_1 = \vec{0} \Rightarrow c_1$ (some non-zero entry) $=0$ $\Rightarrow c_1 = 0$.

If the theorem is true for $k=m$ and we have $m+1$ vectors with distinct eigenvalues, consider

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_m\vec{v}_m + c_{m+1}\vec{v}_{m+1} = \vec{0}$$

Compute $(A-\lambda_1 I)$ times this sum:

$$\Rightarrow c_1\underbrace{(\lambda_1-\lambda_1)}_{0}\vec{v}_1 + \underbrace{c_2(\lambda_2-\lambda_1)\vec{v}_2 + \dots + c_{m+1}(\lambda_{m+1}-\lambda_1)\vec{v}_{m+1}}_{\text{here is a linear combo of } m \text{ vectors } \vec{v}_2 \dots \vec{v}_{m+1} \text{ which have distinct eigenvalues.}} = \vec{0}$$

If the theorem is true for $k=m$ we deduce

$$c_2\underbrace{(\lambda_2-\lambda_1)}_{\neq 0} = 0 = c_3\underbrace{(\lambda_3-\lambda_1)}_{\neq 0} = \dots = c_{m+1}\underbrace{(\lambda_{m+1}-\lambda_1)}_{\neq 0}$$

So $c_2 = c_3 = \dots = c_{m+1} = 0$

so $c_1\vec{v}_1 = 0$, so $c_1 = 0$

thus thm true for $k=1 \Rightarrow$ for $k=2 \Rightarrow$ for $k=3 \dots$ $\Rightarrow$ for every k

<u>Corollary</u> If $A_{n\times n}$ has n different eigenvalues, <u>then</u> A is diagonalizable.

<u>proof</u>. Let $\lambda_1, \lambda_2 \dots \lambda_n$, $\vec{v}_1, \vec{v}_2, \dots \vec{v}_n$ be the "eigenpairs". then $\vec{v}_1, \vec{v}_2, \dots \vec{v}_n$ are independent (by Thm 1), so a basis of $\mathbb{R}^n$ (or $\mathbb{C}^n$). ∎

<u>Corollary</u> (see Theorem 4 page 378)
If the dimension of each eigenspace equals the power k that $(\lambda - \lambda_j)$ appears with in the characteristic polynomial, then A is also diagonalizable (as for $A = \begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix}$), and you get an eigenbasis but taking the union of your biases for each eigenspace

Review Sheet
Math 2250-1 November 14, 2008

Our exam covers chapters 4, 5, EP3.7, and 10 of the text. Only scientific calculators will be allowed on the exam. There is a practice session on this Saturday Nov. 15, in JWB 335, from 11:00-12:30. We will go over a practice exam, which will be posted by Friday at 1:00 p.m.

Chapter 4:

At most 30% of the exam will deal directly with this material....but much of Chapter 5 uses these concepts, so much more than 30% of the exam will be related to chapter 4. (And, as far as matrix computations go, you should remember everything you learned in Chapter 3.)

Know Definitions:

(a) **Vector Space:** A collection of objects which can be added and scalar multiplied, so that the usual arithemetic properties (Page 236) hold. You do not need to memorize all eight of these properties. The key point is that not only is R^n a vector space, but also certain subsets of it are, and so are spaces made out of functions...because functions can be added and scalar multiplied (page 274.)

(b) **Subspace:** a subset of a vector space which is itself of vector space....to check whether a subset is actually a subspace you only have to show that sums and scalar multiples of subset elements are also in the subset (Theorem 1 page 237.) Examples of important subspaces are the solution space to a homogeneous matrix equation (page 239), the span of a collection of vectors (page 243), AND the set of homogenous solutions to a linear differential equation (section 5.2).

(c) A **linear combination** of a set of vectors {v1, v2, ...vn} is any expression c1*v1 + c2*v2 + ... + cn*vn. (page 242)

(d) The **span** of a set of vectors {v1, v2, ...vn} is the collection of all linear combinations. (page 243)

(e) A collection {v1, v2, ...vn} is **linearly dependent** if and only if some linear combinatation (with not all ci's = 0) adds up to the zero vector.

(f) A collection {v1, ... vn} is **linearly independent** if and only if the only linear combination of them which adds up to zero is the one in which all coefficients ci=0. (page 244)

(g) A **basis** for a vector space (or subspace) is a set of vectors {v1, ..., vk} which span the space and which are linearly independent. (page 249.)

(h) The **dimension** of a vector space is the number of elements in any basis.

Know Facts:

(a) If the dimension of a vector space is n, then no collection of fewer than n vectors can span and every collection with more than n elements is dependent.

(b) n vectors in R^n are a basis if and only if the square matrix in which they are the columns is non-singular. So you can use det or rref as a test for basis in this case.

(c) Basically all linear independence and span questions in R^n can be answered using rref. (see below.)

(d) You can toss dependent vectors out of a collection without changing the span. In this manner you can take a spanning set and turn it into a basis.

Do computations:

(a) Be able to check whether vectors are independent or independent, e.g. problems page 248. (4.3) Know how to use rref to check for dependencies.

(b) Be able to find bases for the solution space to homogeneous equations, e.g. problems page 255 (4.4)

Chapter 5 and EP3.7.

At least 50% of the exam will cover this material, and at least 30% of the exam will be from sections 5.4 and 5.5. (Answering questions from 5.4 and 5.5 almost always uses 5.1-5.3 material implicitly.)

5.1-5.3, 5.5 General theory:

Linear differential equations (page 296.)

principle of superposition (e.g. Theorem 1 page 296, also leads to the fact that the general solution y to the inhomogeneous equation is yp + yh, where yp is a particular solution, and yh is the general solution to the homogeneous equation. (Theorem 5 page 306.) Also leads to a method for getting particular solutions which are sums of particular solutions for pieces of the right hand side.)

homogeneous (L(y)=0). Solution space is an n-dimensional vector space. Know how to find it for constant coefficients, using exponentials and the resulting **characteristic equation** and **Euler formula** if necessary (section 5.3 and problems). What to do with repeated roots. The Wronskian test for linear independence .

nonhomogeneous (L(y)=f). Know how to find particular solutions by the method of undetermined coefficients. (Section 5.5 and problems) (But variation of parameters will not be on the exam.)

initial value problem, existence and uniqueness. Know how to solve initial value problems by finding yp, and yh, and then finding values of constants in yh to match initial conditions.

5.4, 5.6, EP3.7 Mechanical vibrations and forced oscillations; electrical circuit analog

unforced oscillations (i.e. solutions to the homogeneous DE):

undamped (simple harmonic motion)

going from A*cos(wt) + B*cos(wt) to C*cos(wt-a). (The ABC triangle, amplitude and phase.)
derivation of spring equation from Newton's and Hooke's Laws.

damped.

under-damped, over-damped, critically damped. Know how to recognize, and different forms of the solution.

forced oscillations:

undamped:

resonance, and when it arises. form of solution, as follows from general theory above.

beating, when w is close to w0.

damped:

general solution is sum of **steady state periodic**, with **transient**. How to find each piece, and express the steady state periodic solution in amplitude- phase form.

practical resonance, will occur if damping is small and driving frequency is near natural frequency.

An RLC circuit corresponds to a mass-spring configuration, where L corresponds to mass, R to damping coefficient, and 1/C to spring constant.

Chapter 10:

At least 30% of the exam will deal directly with this material. The most efficient way for me to test it is to have you solve IVPs from chapter 5, using chapter 5 techniques as well as Laplace transform techniques. You will be responsible for 10.1-10.4 on the exam (not 10.5 which we surveyed quickly and in which you worked several homework problems). You will be provided with the Laplace transform table on the front book cover. The one thing that table is missing is the definition of Laplace transform, which you should memorize (page 568).

Key skills:

Using the definition of Laplace transform to reproduce a table entry or to compute the transform of a function not
on the table.

Converting an initial value problem into an algebraic equation for the Laplace transform of the solution (section 10.2).

Using the table to convert in both directions between functions and their Laplace transforms. (covers 10.1-10.5 facts)

Setting up and using partial fractions (and possibly **completing the square**) to simplify Laplace transforms so that you can use the table to invert them. (section 10.3)

Using and computing convolutions to invert the product of two Laplace transforms (section 10.4).

Using the **unit step function** to turn forcing functions on and off in spring configuration DEs (section 10.5) is important, but won't be on the exam.