

Math 2250-1  
Wednesday Nov. 12

HW for Friday Nov. 21

①

- 6.1 7, 17, 18, 24, 29
- 6.2 4, 10, 19 28
- 6.3 5, 15, 26 31 32 41

Our second exam is this coming Monday, Nov. 17  
It will cover chapters 4.1-4.4, 4.7,  
5.1-5.6, EP 3.7  
10.1-10.5

I'll post a review sheet & practice exam some time Thursday - do we want a problem session Friday pm or Saturday?

96.1 A return to linear algebra, so we can tackle the next step in DE's:

1<sup>st</sup> order DE's

n<sup>th</sup> order linear DE's

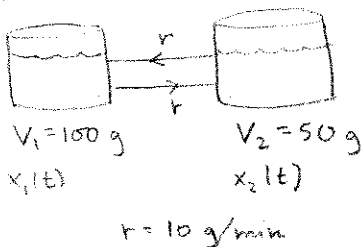
in particular, springs

Laplace transform

systems of 1<sup>st</sup> & 2<sup>nd</sup> order DE's

e.g. tank systems & spring systems

example:



$$\frac{dx_1}{dt} = -10 \frac{x_1}{100} + 10 \frac{x_2}{50}$$

$$\frac{dx_2}{dt} = 10 \frac{x_1}{100} - 10 \frac{x_2}{50}$$

by the way, could you solve this via Laplace transform?

in matrix form, and IVP:

$$\begin{cases} \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -0.1 & 0.2 \\ 0.1 & -0.2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \end{cases}$$

This is an example of a 1<sup>st</sup> order, constant coefficient, homogeneous linear system of DE's.

$$\begin{cases} \vec{x}'(t) = A\vec{x}(t) \\ \vec{x}(t_0) = \vec{x}_0 \end{cases}$$

Solution space will be  $n$ -dim'l, and we will look for basis fns of the form  $x(t) = e^{\lambda t} \vec{v}$  (where  $\vec{v}$  is a constant vector)

plug it in to  $\vec{x}' = A\vec{x}$ :

$$\lambda e^{\lambda t} \vec{v} = A e^{\lambda t} \vec{v} = e^{\lambda t} A \vec{v}$$

divide by  $e^{\lambda t}$ :  $\lambda \vec{v} = A \vec{v}$  ←  $\vec{v}$  is an eigenvector of  $A$  with eigenvalue  $\lambda$

(2)

In tank example  $A = \begin{bmatrix} -.1 & .2 \\ .1 & -.2 \end{bmatrix}$

Note  $\begin{bmatrix} -.1 & .2 \\ .1 & -.2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\begin{bmatrix} -.1 & .2 \\ .1 & -.2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -.3 \\ .3 \end{bmatrix} = -.3 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \lambda = 0, \text{ soltn}$   
 $1 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \lambda = -.3, \text{ soltn}$   
 $e^{-.3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Notice the solutions to the homogeneous system

$$\frac{d\vec{x}}{dt} = A\vec{x}$$

form a subspace

check:

what happens as  $t \rightarrow \infty$ ?

so  $\vec{x}(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-.3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$  solves tank problem.

to solve IVP:  $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ , solve the system  $\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$  ■

How to find eigenvalues and eigenvectors?

$A\vec{v} = \lambda\vec{v}$  ( $\vec{v} \neq \vec{0}$  is interesting case)

$A\vec{v} - \lambda\vec{v} = \vec{0}$

$(A - \lambda I)\vec{v} = \vec{0}$

↑

in order for there to be nontrivial solutions to this matrix eqn, the matrix  $A - \lambda I$  must be singular (non-invertible)

iff  $\det(A - \lambda I) = 0$

↑  
 polynomial in  $\lambda$ , called characteristic polynomial

Algorithm for finding evals/evecs:

- ① Find roots of characteristic polynomial
- ② For each root,  $\lambda$  find an eigenspace basis, i.e. basis for soltn space to  $(A - \lambda I)\vec{v} = \vec{0}$ .

Example  $A = \begin{bmatrix} -1 & .2 \\ .1 & -2 \end{bmatrix}$

$$|A - \lambda I| = \begin{vmatrix} -1-\lambda & .2 \\ .1 & -2-\lambda \end{vmatrix} = (\lambda + 1)(\lambda + 2) - .02$$

$$= \lambda^2 + .3\lambda + .02 - .02$$

$$= \lambda(\lambda + .3)$$

$\lambda = 0, -.3$

$\lambda = 0$  espace:

$$\begin{array}{cc|c} -1 & .2 & 0 \\ .1 & -2 & 0 \\ \hline -1 & 2 & 0 \\ 0 & 0 & 0 \end{array}$$

$v_2 = t$   
 $v_1 = 2t$   
 $\vec{v} = t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

basis  $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$

$\lambda = -.3$  espace

$$\begin{array}{cc|c} .2 & .2 & 0 \\ .1 & -1 & 0 \\ \hline 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}$$

$v_2 = t$   
 $v_1 = -t$   
 $\vec{v} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

basis  $\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$

Example  $A = \begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix}$

$\lambda(-\lambda^2 + 7\lambda - 12)$   
 $\lambda(\lambda - 3)(4 - \lambda)$

$$|A - \lambda I| = \begin{vmatrix} 4-\lambda & -2 & 1 \\ 2 & -\lambda & 1 \\ 2 & -2 & 3-\lambda \end{vmatrix} = (4-\lambda)(-\lambda)(3-\lambda) - 4 - 4 + 2\lambda + 2(4-\lambda) + 4(3-\lambda)$$

$$= \lambda^3(-1) + \lambda^2(7) + \lambda(-12 + 2 - 2 - 4) + 1(12)$$

$$= -\lambda^3 + 7\lambda^2 - 16\lambda + 12$$

possible integer roots:  $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6$

$p(2) = -8 + 28 - 32 + 12 = 0!$

so  $\lambda - 2$  is a factor

$$\lambda - 2 \overline{) \begin{array}{r} -\lambda^3 + 7\lambda^2 - 16\lambda + 12 \\ -\lambda^3 + 2\lambda^2 \\ \hline 5\lambda^2 - 16\lambda \\ 5\lambda^2 - 10\lambda \\ \hline -6\lambda + 12 \\ -6\lambda + 12 \\ \hline 0 \end{array}}$$

$p(\lambda) = (\lambda - 2)(-1)(\lambda^2 - 5\lambda + 6)$   
 $= (\lambda - 2)(\lambda - 2)(\lambda - 3)$   
 $= (\lambda - 2)^2(\lambda - 3)$

$\lambda = 2$  espace:

$$\begin{array}{ccc|c} 2 & -2 & 1 & 0 \\ 2 & -2 & 1 & 0 \\ 2 & -2 & 1 & 0 \\ \hline 2 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 1 & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}$$

$v_3 = t$   
 $v_2 = s$   
 $v_1 = s - \frac{1}{2}t$

$$\vec{v} = s \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

basis  $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \right\}$

$\lambda = 3$  espace

$$\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 2 & -3 & 1 & 0 \\ 2 & -2 & 0 & 0 \\ \hline 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 2 & -2 & 0 \\ \hline 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}$$

$-2R_1 + R_2$   
 $-2R_1 + R_3$   
  
 $2R_2 + R_1$

$v_3 = t$   
 $v_2 = t$   
 $v_1 = t$

$\vec{v} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$   
 basis  $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ .

it will be good, for solving our systems of 1st order homog DE's, if we can amalgamate our eigenspace bases to get a basis for  $\mathbb{R}^n$ . That happens in this example, since

$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$  is an  $\mathbb{R}^3$  basis.  
 Why?

[Such matrices A will be called "diagonalizable"].