Math 2250-1

November 11, 2008

Guess the resonance game, using convolution formula, section 10.4

```
> with(plots):with(inttrans):
    #the library inttrans includes Laplace
```

We are considering the undamped forced harmonic oscillator

$$x''(t) + x(t) = f(t)$$

with initial data x(0)=v(0)=0. When we take the Laplace transform of this equation we deduce

$$X(s) = \frac{1\ F(s)}{s^2 + 1}$$

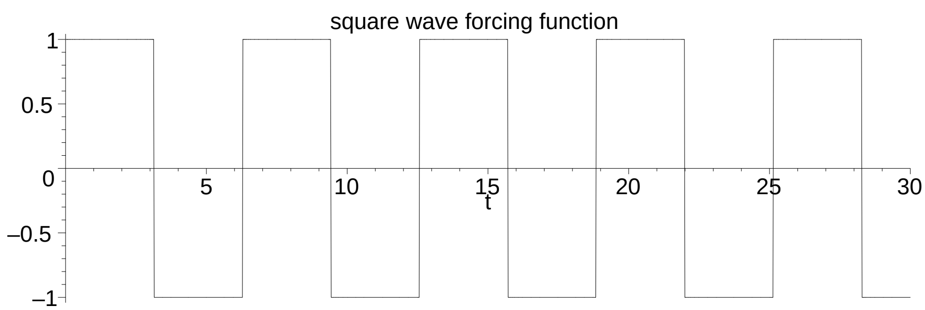
so that the convolution theorem implies $x(t) = \sin * f(t)$. Since the unforced system has a natural angular frequency $\omega_0 = 1$, we expect resonance when the forcing function has the corresponding period of $\frac{2\pi}{w_0} = 2\pi$. We will discover that there is a surprising possible error in our reasoning.

Example 1: A square wave forcing function with amplitude 1 and period 2π. Let's talk about how we came up with the formula (which works until $t = 11\pi$).

```
> f:=t->-1+2*sum((-1)^n*Heaviside(t-n*Pi),n=0..10);
   #Heaviside was an early user of the unit step function
   #and so Maple names it after him
```

$$f := t \to -1 + 2\left(\sum_{n=0}^{10} (-1)^n \text{Heaviside}(t - n\pi)\right)$$

```
> plot(f(t),t=0..30,color=black,title=`square wave forcing
  function`);
```




```
> x:=t->int(sin(t-tau)*f(tau),tau=0..t);
     #convolution formula for the solution
```

$$x := t \to \int_0^t \sin(t - \tau)\, f(\tau)\, d\tau$$

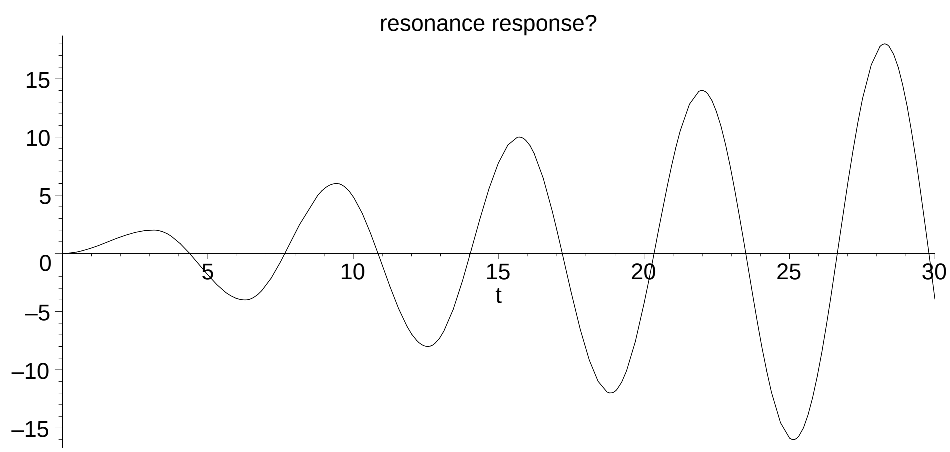
You actually could use the convolution formula to work out x(t) by hand, if you wished to do so!

```
> x(t);  #you actually could work this out by hand!
```

$$-1-2\,\mathrm{Heaviside}(t-3\pi)+2\,\mathrm{Heaviside}(t)-2\,\mathrm{Heaviside}(t-3\pi)\cos(t)-2\,\mathrm{Heaviside}(t)\cos(t)$$
$$+2\,\mathrm{Heaviside}(t-4\pi)-2\,\mathrm{Heaviside}(t-\pi)-2\,\mathrm{Heaviside}(t-4\pi)\cos(t)-2\,\mathrm{Heaviside}(t-\pi)\cos(t)$$
$$-2\,\mathrm{Heaviside}(t-5\pi)+2\,\mathrm{Heaviside}(t-2\pi)-2\,\mathrm{Heaviside}(t-5\pi)\cos(t)$$
$$-2\,\mathrm{Heaviside}(t-2\pi)\cos(t)+2\,\mathrm{Heaviside}(t-6\pi)-2\,\mathrm{Heaviside}(t-9\pi)\cos(t)$$
$$-2\,\mathrm{Heaviside}(t-6\pi)\cos(t)+2\,\mathrm{Heaviside}(t-10\pi)-2\,\mathrm{Heaviside}(t-7\pi)$$
$$-2\,\mathrm{Heaviside}(t-10\pi)\cos(t)-2\,\mathrm{Heaviside}(t-7\pi)\cos(t)+2\,\mathrm{Heaviside}(t-8\pi)$$
$$-2\,\mathrm{Heaviside}(t-8\pi)\cos(t)-2\,\mathrm{Heaviside}(t-9\pi)+\cos(t)$$

As expected (?), we get resonance.

```
> plot(x(t),t=0..30,color=black,title=‘resonance response?‘);
```

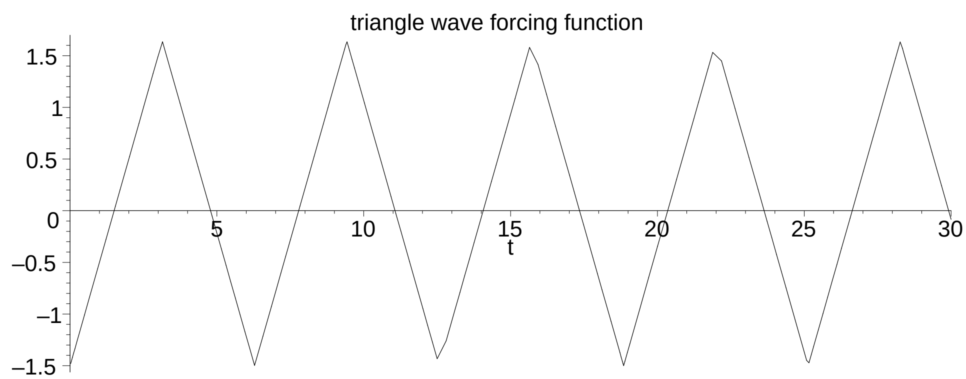


Example 2: triangle wave forcing function, same period.

```
> g:=t->int(f(u),u=0..t)-1.5;
  #this should be a triangle wave...
```

$$g := t \to \int_0^t \mathrm{f}(u)\, du - 1.5$$

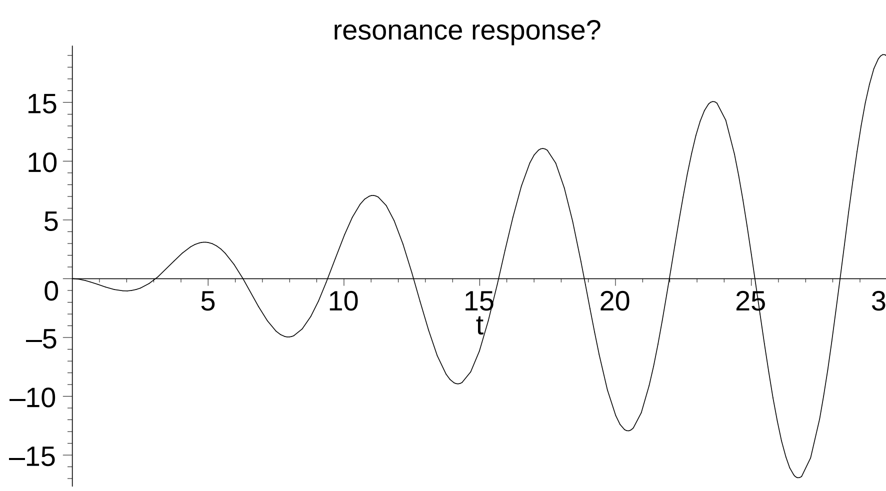
```
> plot(g(t),t=0..30,color=black, title=‘triangle wave forcing
  function‘);
```




```
> y:=t->int(sin(t-tau)*g(tau),tau=0..t);
```

$$y := t \to \int_0^t \sin(t - \tau)\, \mathrm{g}(\tau)\, d\tau$$

```
> plot(y(t),t=0..30,color=black, title=‘resonance response?‘);
```

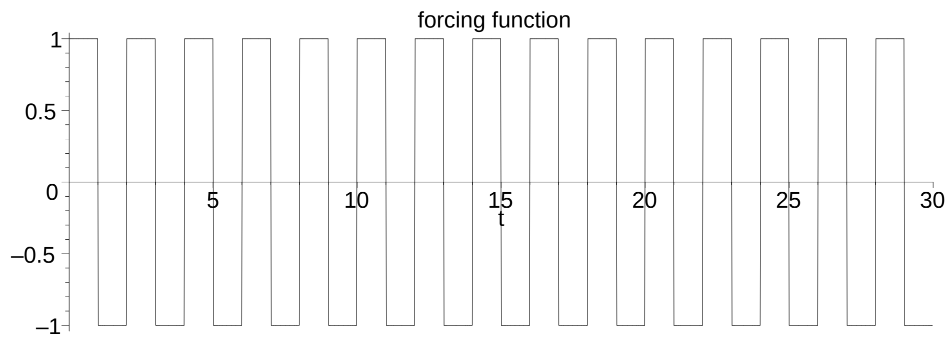


Example 3: Now let's force with a period which is not the natural one. This square wave has period 2.

```
> h:=t->-1+2*sum((-1)^n*Heaviside(t-n),n=0..30);
```

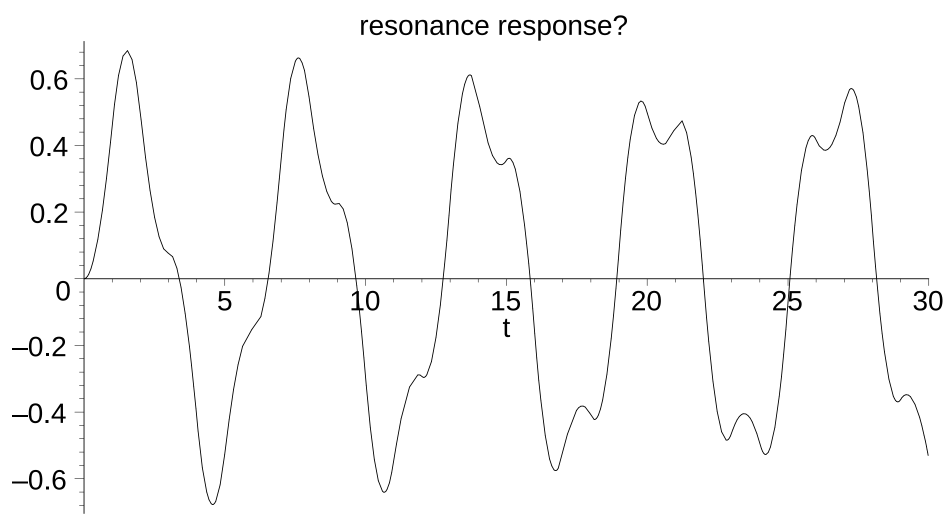
$$h := t \to -1 + 2\left(\sum_{n=0}^{30} (\text{-}1)^n \,\text{Heaviside}(t-n)\right)$$

```
> plot(h(t),t=0..30,color=black, title='forcing function');
```




```
> z:=t->int(sin(t-tau)*h(tau),tau=0..t);
  plot(z(t),t=0..30,color=black,title='resonance response?');
```

$$z := t \to \int_0^t \sin(t-\tau)\,\text{h}(\tau)\,d\tau$$

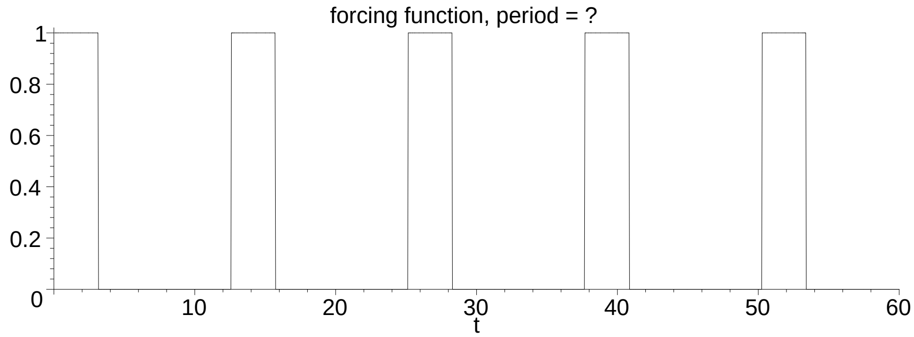


Example 4: A square wave which does not have the natural period, so we don't expect resonance?

```
> k:=t->sum(Heaviside(t-4*Pi*n)-Heaviside(t-4*Pi*n-Pi),
   n=0..5);
```

$$k := t \to \sum_{n=0}^{5} (\text{Heaviside}(t - 4\,n\,\pi) - \text{Heaviside}(t - 4\,n\,\pi - \pi))$$

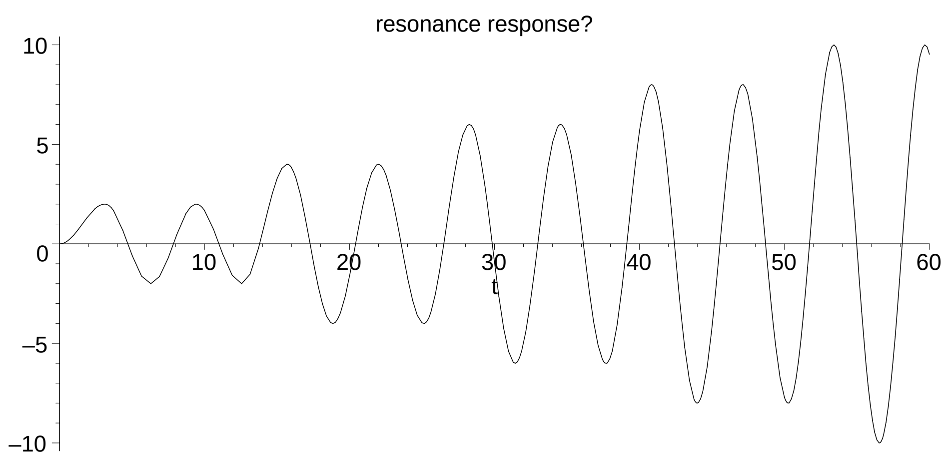
```
> plot(k(t),t=0..60,color=black,title='forcing function, period =
  ?');
>
```




```
> w:=t->int(sin(t-tau)*k(tau),tau=0..t);
```

$$w := t \to \int_0^t \sin(t - \tau)\,\text{k}(\tau)\,d\tau$$

```
> plot(w(t),t=0..60,color=black,title='resonance response?');
```



Hey, what happened???? How do we need to modify our thinking if we force a system with something which is not sinusoidal, in terms of worrying about resonance?