Math 2250-1

Tuesday Nov. 11

§5.4-5.4

1^{st} do on-off problem, last example Monday

Then convolution!!

convolution (for Laplace transf.)

$$f*g(t) := \int_0^t f(\tau)g(t-\tau)\,d\tau$$
$$= g*f(t)$$

<u>Theorem</u>

$$\mathcal{L}\{f*g(t)\}(s) = F(s)G(s)$$

<u>example</u>: Verify the theorem

for $f(t) = \sin t$

$g(t) = \cos t$

[you may need the trig identity $\sin^2\tau = \frac{1-\cos 2\tau}{2}$]

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

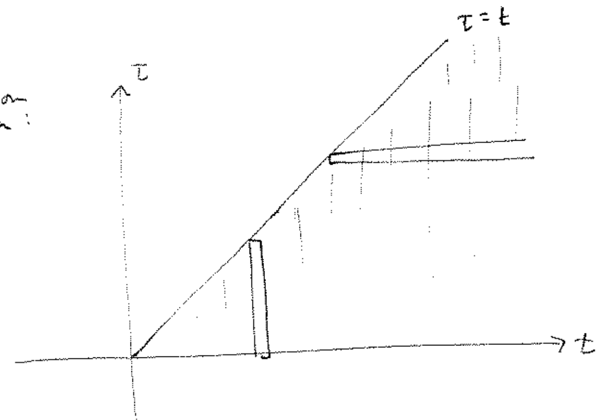
$f(t)$	$F(s)$	
$c_1 f_1(t) + c_2 f_2(t)$	$c_1 F_1(s) + c_2 F_2(s)$	$\mathcal{L}$ is linear!
1	$1/s$	
e^{at}	$1/(s-a)$	$\mathrm{Re}\, s > \mathrm{Re}\, a$
$f'(t)$	$sF(s) - f(0)$	
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$	
$f'''(t)$	$s^3F(s) - s^2f(0) - sf'(0) - f''(0)$	
	etc	
$\int_0^t f(\tau)\,d\tau$	$\frac{F(s)}{s}$	
$t f(t)$	$-F'(s)$	
$t^2 f(t)$	$F''(s)$	
$t^3 f(t)$	$-F'''(s)$	
$f(t)/t$	$\int_s^\infty F(\sigma)\,d\sigma$	
	etc	
$u(t-a)$	e^{-as}/s	
$u(t-a)f(t-a)$	$e^{-as}F(s)$	
$e^{at}f(t)$	$F(s-a)$	
1	$1/s$	
t	$1/s^2$	
t^2	$2/s^3$	
t^n	$n!/s^{n+1}$	
$\cos kt$	$s/(s^2+k^2)$	
$\sin kt$	$k/(s^2+k^2)$	
$\cosh kt$	$s/(s^2-k^2)$	
$\sinh kt$	$k/(s^2-k^2)$	
$e^{at}\cos kt$	$\frac{s-a}{(s-a)^2+k^2}$	
$e^{at}\sin kt$	$\frac{k}{(s-a)^2+k^2}$	
$t^n e^{at}$	$n!/(s-a)^{n+1}$	
$\frac{1}{2k^3}(\sin kt - kt\cos kt)$	$\frac{1}{(s^2+k^2)^2}$	
$\frac{t}{2k}\sin kt$	$\frac{s}{(s^2+k^2)^2}$	
$t\cos kt$	$(s^2-k^2)/(s^2+k^2)^2$	
$(f*g)(t)$	$F(s)G(s)$	today!

proof of convolution theorem:
(is a good review of iterated integrals)

$$\mathcal{L}\{f*g\}(s) = \int_0^\infty e^{-st}\left(\int_0^t f(\tau)\,g(t-\tau)\,d\tau\right) dt$$

$$= \int_0^\infty \int_0^t e^{-st} f(\tau)\,g(t-\tau)\,d\tau\,dt$$

integration region:



interchange limits:

$$= \int_0^\infty \int_\tau^\infty e^{-st} f(\tau)\,g(t-\tau)\,dt\,d\tau$$

$$= \int_0^\infty \int_\tau^\infty e^{-s\tau} f(\tau)\, e^{-s(t-\tau)} g(t-\tau)\,dt\,d\tau \qquad \text{(pattern recognition)}$$

$$= \int_0^\infty e^{-s\tau} f(\tau) \left[\int_\tau^\infty e^{-s(t-\tau)} g(t-\tau)\,dt \right] d\tau$$

$\tilde{t} = t - \tau$
$d\tilde{t} = dt$

$$\underbrace{\left[\int_0^\infty e^{-s\tilde{t}} g(\tilde{t})\,d\tilde{t} \right]}_{G(s)}$$

$$= G(s) \int_0^\infty e^{-s\tau} f(\tau)\,d\tau$$

$$= G(s)\,F(s) \qquad !!$$

Application: for any nonhomogeneous, const. coeff. linear DE or system, there is a convolution formula for the solution (must be equivalent to variation of parameters ?!)

example

$$\begin{cases} mx'' + cx' + kx = f(t) \\ x(0) = x_0 \\ x'(0) = v_0 \end{cases}$$

$\mathcal{L}$: $X(s)\left[ms^2 + cs + k\right] = F(s) + (sx_0 + v_0)m + x_0 c$

$$X(s) = \frac{F(s)}{ms^2+cs+k} + \frac{(sx_0+v_0)m + x_0 c}{ms^2+cs+k}$$

↑ use convolution formula to invert

↑ table

subexample

$$\begin{cases} x'' + x = f(t) \\ x(0) = 0 \\ x'(0) = 0 \end{cases}$$

$\mathcal{L}$: $X(s)(s^2+1) = F(s)$

$$X(s) = \frac{1}{s^2+1} F(s)$$

$$x(t) = (\sin * f)(t) = \int_0^t f(\tau)\sin(t-\tau)\,d\tau = \int_0^t (\sin\tau) f(t-\tau)\,d\tau$$

see handout & play the resonance game.

Math 2250-1
November 11, 2008
Guess the resonance game, using convolution formula, section 10.4

```
> with(plots):with(inttrans):
    #the library inttrans includes Laplace
```

We are considering the undamped forced harmonic oscillator

$$x''(t) + x(t) = f(t)$$

with initial data x(0)=v(0)=0. When we take the Laplace transform of this equation we deduce

$$X(s) = \frac{1\,F(s)}{s^2+1}$$

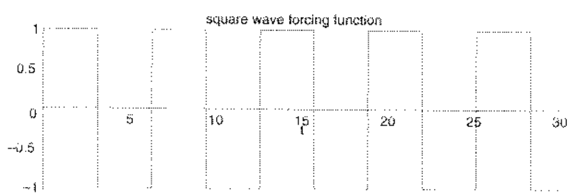
so that the convolution theorem implies $x(t) = \sin * f(t)$. Since the unforced system has a natural angular frequency $\omega_0 = 1$, we expect resonance when the forcing function has the corresponding period of $\frac{2\pi}{w_0} = 2\pi$. We will discover that there is a surprising possible error in our reasoning.

Example 1: A square wave forcing function with amplitude 1 and period 2π. Let's talk about how we came up with the formula (which works until $t = 11\pi$).

```
> f:=t->-1+2*sum((-1)^n*Heaviside(t-n*Pi),n=0..10);
   #Heaviside was an early user of the unit step function
   #and so Maple names it after him
```

$$f := t \to -1 + 2\left(\sum_{n=0}^{10} (-1)^n \,\text{Heaviside}(t - n\pi)\right)$$

```
> plot(f(t),t=0..30,color=black,title='square wave forcing
  function');
```




```
> x:=t->int(sin(t-tau)*f(tau),tau=0..t);
     #convolution formula for the solution
```

$$x := t \to \int_0^t \sin(t-\tau)\, f(\tau)\, d\tau$$

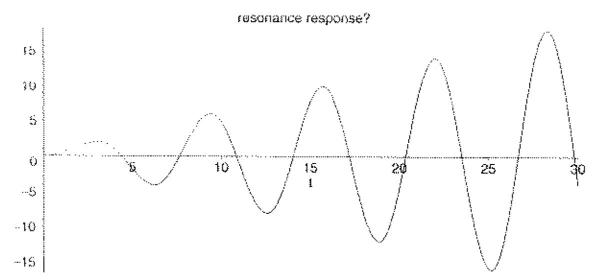
You actually could use the convolution formula to work out x(t) by hand, if you wished to do so!

```
> x(t);  #you actually could work this out by hand!
```

$$-1 - 2\,\text{Heaviside}(t-3\pi) + 2\,\text{Heaviside}(t) - 2\,\text{Heaviside}(t-3\pi)\cos(t) - 2\,\text{Heaviside}(t)\cos(t)$$
$$+ 2\,\text{Heaviside}(t-4\pi) - 2\,\text{Heaviside}(t-\pi) - 2\,\text{Heaviside}(t-4\pi)\cos(t) - 2\,\text{Heaviside}(t-\pi)\cos(t)$$
$$- 2\,\text{Heaviside}(t-5\pi) + 2\,\text{Heaviside}(t-2\pi) - 2\,\text{Heaviside}(t-5\pi)\cos(t)$$
$$- 2\,\text{Heaviside}(t-2\pi)\cos(t) + 2\,\text{Heaviside}(t-6\pi) - 2\,\text{Heaviside}(t-9\pi)\cos(t)$$
$$- 2\,\text{Heaviside}(t-6\pi)\cos(t) + 2\,\text{Heaviside}(t-10\pi) - 2\,\text{Heaviside}(t-7\pi)$$
$$- 2\,\text{Heaviside}(t-10\pi)\cos(t) - 2\,\text{Heaviside}(t-7\pi)\cos(t) + 2\,\text{Heaviside}(t-8\pi)$$
$$- 2\,\text{Heaviside}(t-8\pi)\cos(t) - 2\,\text{Heaviside}(t-9\pi) + \cos(t)$$

As expected (?), we get resonance.

```
> plot(x(t),t=0..30,color=black,title='resonance response?');
```

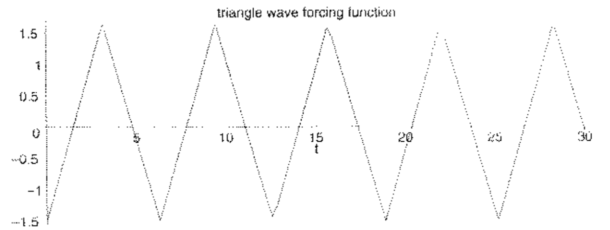


Example 2: triangle wave forcing function, same period.

```
> g:=t->int(f(u),u=0..t)-1.5;
  #this should be a triangle wave...
```

$$g := t \to \int_0^t \mathrm{f}(u)\,du - 1.5$$

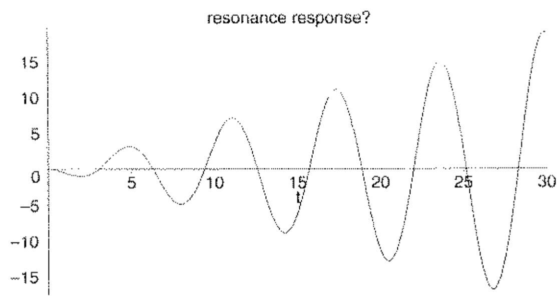
```
> plot(g(t),t=0..30,color=black, title='triangle wave forcing
  function');
```




```
> y:=t->int(sin(t-tau)*g(tau),tau=0..t);
```

$$y := t \to \int_0^t \sin(t-\tau)\,\mathrm{g}(\tau)\,d\tau$$

```
> plot(y(t),t=0..30,color=black, title='resonance response?');
```

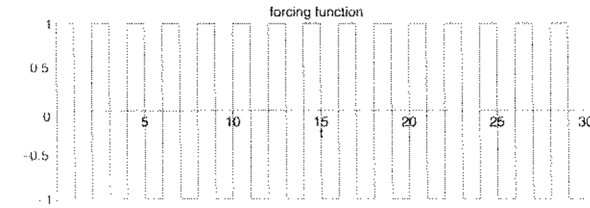


Example 3: Now let's force with a period which is not the natural one. This square wave has period 2.

```
> h:=t->-1+2*sum((-1)^n*Heaviside(t-n),n=0..30);
```

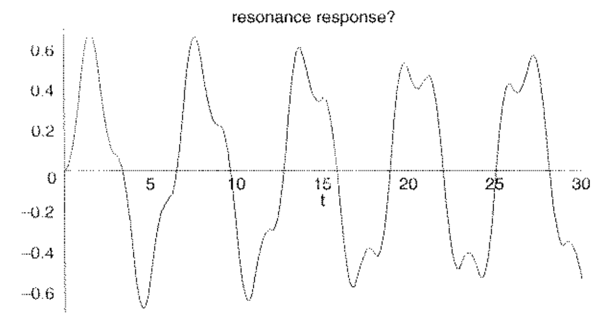
$$h := t \to -1 + 2\left(\sum_{n=0}^{30} (-1)^n \,\mathrm{Heaviside}(t-n)\right)$$

```
> plot(h(t),t=0..30,color=black, title='forcing function');
```




```
> z:=t->int(sin(t-tau)*h(tau),tau=0..t);
  plot(z(t),t=0..30,color=black,title='resonance response?');
```

$$z := t \to \int_0^t \sin(t-\tau)\,\mathrm{h}(\tau)\,d\tau$$

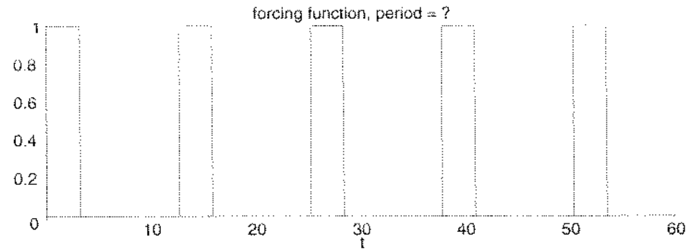


Example 4: A square wave which does not have the natural period, so we don't expect resonance?

```
> k:=t->sum(Heaviside(t-4*Pi*n)-Heaviside(t-4*Pi*n-Pi),
   n=0..5);
```

$$k := t \to \sum_{n=0}^{5} (\mathrm{Heaviside}(t - 4\,n\,\pi) - \mathrm{Heaviside}(t - 4\,n\,\pi - \pi))$$

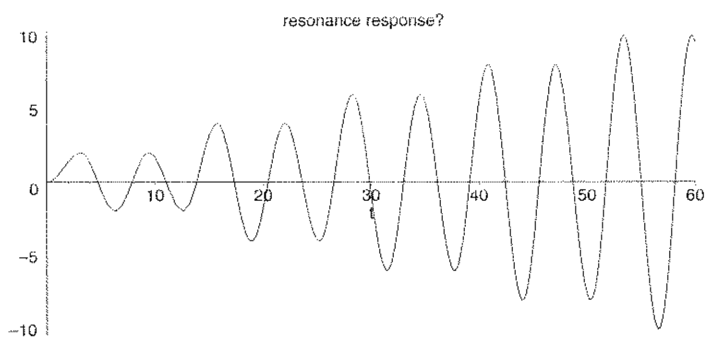
```
> plot(k(t),t=0..60,color=black,title='forcing function, period =
  ?');
>
```




```
> w:=t->int(sin(t-tau)*k(tau),tau=0..t);
```

$$w := t \to \int_0^t \sin(t - \tau)\,\mathrm{k}(\tau)\,d\tau$$

```
> plot(w(t),t=0..60,color=black,title='resonance response?');
```



Hey, what happened???? How do we need to modify our thinking if we force a system with something which is not sinusoidal, in terms of worrying about resonance?