Math 2250-1
Monday Nov. 10
Laplace transform continued

I've added to your HW for Friday: 10.5 7, 13 31, 34

today: First to the example on page 3 Friday - partial fractions!!
Then:

$\mathcal{L}\{t\cos kt\}$:

$$\mathcal{L}\{\cos kt\}(s) = \frac{s}{s^2+k^2}$$

so $\mathcal{L}\{t\cos kt\} = -F'(s)$

$$= -\frac{1}{s^2+k^2} - s(-1)(s^2+k^2)^{-2}(2s)$$

$$= \frac{-s^2-k^2+2s^2}{(s^2+k^2)^2} = \frac{s^2-k^2}{(s^2+k^2)^2} \checkmark$$

$\mathcal{L}\{t\sin kt\}$:

$$\mathcal{L}\{\sin kt\}(s) = \frac{k}{s^2+k^2}$$

so $\mathcal{L}\{t\sin kt\} = -F'(s)$

$$= -k(-1)(s^2+k^2)^{-2}2s$$

$$= \frac{+2ks}{(s^2+k^2)^2} \checkmark$$

So $\mathcal{L}^{-1}\left(\frac{1}{(s^2+k^2)^2}\right) = \mathcal{L}^{-1}\left\{\left\{\frac{s^2+k^2}{(s^2+k^2)^2} - \frac{(s^2-k^2)}{(s^2+k^2)^2}\right\}\frac{1}{2k^2}\right\}(t)$

$$= \frac{1}{2k^2}\left(\frac{1}{k}\sin kt - t\cos kt\right) \checkmark$$

(useful in resonance problems).

$x(t)$ / $f(t)$	$X(s)$ / $F(s)$
1	$1/s$
t	$1/s^2$
t^n	$n!/s^{n+1}$ $n\in\mathbb{N}$
e^{at}	$1/(s-a)$
$\cos kt$	$s/(s^2+k^2)$
$\sin kt$	$k/(s^2+k^2)$
$\cosh kt$	$s/(s^2-k^2)$
$\sinh kt$	$k/(s^2-k^2)$
$f'(t)$	$sF(s)-f(0)$
$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
etc	
$\int_0^t f(\tau)d\tau$	$\frac{F(s)}{s}$
analogous	
$tf(t)$	$-F'(s)$
$t^2f(t)$	$F''(s)$
etc	
$\frac{f(t)}{t}$	$\int_s^\infty F(\sigma)d\sigma$
$t\cos kt$	$(s^2-k^2)/(s^2+k^2)^2$
today $\frac{1}{2k}t\sin kt$	$\frac{s}{(s^2+k^2)^2}$
$\frac{1}{2k^3}(\sin kt - kt\cos kt)$	$\frac{1}{(s^2+k^2)^2}$
te^{at}	$\frac{1}{(s-a)^2}$
page 3 today: $u(t-a)$	e^{-as}/s
$u(t-a)f(t-a)$	$e^{-as}F(s)$
$e^{at}f(t)$	$F(s-a)$
$e^{at}\cos kt$	$\frac{s-a}{(s-a)^2+k^2}$
$e^{at}\sin kt$	$\frac{k}{(s-a)^2+k^2}$
$f*g(t)$	$F(s)G(s)$ ⇐ tomorrow!!

Example p. 595

$$\begin{cases} x'' + \omega_0^2 x = F_0 \sin \omega t \\ x(0) = 0 \\ x'(0) = 0 \end{cases}$$

resonance (or not) revisited

$\mathcal{L}$:

$$s^2 X(s) - 0s - 0 + \omega_0^2 X(s) = F_0 \frac{\omega}{s^2+\omega^2}$$

$$X(s)\left[s^2+\omega_0^2\right] = F_0 \frac{\omega}{s^2+\omega^2}$$

$$X(s) = F_0 \left(\frac{\omega}{s^2+\omega^2}\right)\left(\frac{1}{s^2+\omega_0^2}\right)$$

$\omega \neq \omega_0$

$$X(s) = F_0 \omega \left[\frac{1}{s^2+\omega^2} - \frac{1}{s^2+\omega_0^2}\right]\left[\frac{1}{\omega_0^2-\omega^2}\right]$$

so $$x(t) = \frac{F_0 \omega}{\omega_0^2-\omega^2}\left[\frac{1}{\omega}\sin \omega t - \frac{1}{\omega_0}\sin \omega_0 t\right]$$

much easier than
method of undetermined coeffs!

$\omega = \omega_0$ resonance

$$X(s) = \frac{F_0 \omega_0}{(s^2+\omega_0^2)^2}$$

$x(t) =$ (from table!)

$$= \frac{F_0 \omega_0}{2\omega_0^3}\left(\sin \omega_0 t - \omega_0 t \cos \omega_0 t\right)$$

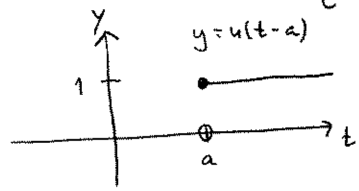
$$x(t) = \frac{F_0}{2\omega_0^2}\left(\sin \omega_0 t - \omega_0 t \cos \omega_0 t\right)$$

resonance!

The unit step function:

$$u(t) := \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases} \qquad \text{thus} \quad u(t-a) = \begin{cases} 1, & t-a \geq 0 \quad (\underline{t \geq a}) \\ 0 & t-a < 0 \quad (t < a) \end{cases}$$

"Heaviside" in Maple



$$\mathcal{L}\{u(t-a)\}(s) = \int_0^\infty e^{-st} u(t-a)\,dt = \underbrace{\int_0^a \cancel{0}}_{0} + \int_a^\infty e^{-st}\,dt = \left. \frac{e^{-st}}{-s} \right]_a^\infty = 0 - \frac{e^{-as}}{-s} \quad (s>0)$$

$$= \boxed{\frac{e^{-as}}{s}} \qquad \underline{a > 0}$$

to "turn on" a function at time $t = a^+$, consider $u(t-a)f(t-a)$

$$\mathcal{L}\{u(t-a)f(t-a)\}s = \int_0^\infty e^{-st} u(t-a) f(t-a)\,dt = \int_a^\infty e^{-st} f(t-a)\,dt = \int_0^\infty e^{-s(\tilde{t}+a)} f(\tilde{t})\,d\tilde{t}$$

subs $\tilde{t} = t - a$, $d\tilde{t} = dt$, $t = \tilde{t} + a$; $t = a$, $\tilde{t} = 0$; $t = \infty$, $\tilde{t} = \infty$

$$\| $$

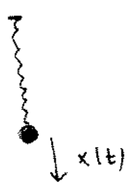
$$e^{-sa} \int_{\tilde{t}=0}^\infty e^{-s\tilde{t}} f(\tilde{t})\,d\tilde{t}$$

$$\| $$

$$\boxed{e^{-as} F(s)}$$

<u>Example</u> (p 609)

$m = 32$ lb ($m = 1$ slug), $c = 0$,



$k = 4$ lb/ft. mass initially at equilibrium ($x(0) = 0$, $x'(0) = 0$)

at $t = 0$ apply $F(t) = \cos 2t$

at $t = 2\pi$ force is turned off. Solve IVP

$$\begin{cases} x'' + 4x = F(t) \\ x(0) = 0 \\ x'(0) = 0 \end{cases}$$

<u>soln</u>

$$s^2 X(s) - 0 - 0 + 4X(s) = F(s)$$

$$X(s)(s^2+4) = F(s)$$

$$f(t) = \cos 2t\,(1 - u(t-2\pi))$$
$$= \cos 2t - \cos 2t\; u(t-2\pi)$$
$$= \cos 2t - \cos(2(t-2\pi))\, u(t-2\pi)$$

$$F(s) = \frac{s}{s^2+4} - e^{-2\pi s} \frac{s}{s^2+4}$$

$$X(s) = \frac{1}{s^2+4}\left(\frac{s}{s^2+4} - e^{-2\pi s}\frac{s}{s^2+4}\right)$$

$$= \frac{s}{(s^2+4)^2} - e^{-2\pi s} \frac{s}{(s^2+4)^2}$$

Table!

$$x(t) = \frac{1}{4} t \sin 2t - u(t-2\pi)\frac{1}{4}(t-2\pi)\sin 2t$$

$$= \begin{cases} \frac{t}{4}\sin 2t & 0 \leq t < 2\pi \\ \frac{\pi}{2}\sin 2t & t \geq 2\pi \end{cases}$$

```
> with(plots):
> plot1:=plot(.25*t*sin(2*t)-Heaviside(t-2*Pi)*(t-2*Pi)/4*sin(2*t),
      t=0..15,color=black):
  plot2:=plot(t/4,t=0..2*Pi,color=black,linestyle=2):
  plot3:=plot(-t/4,t=0..2*Pi,color=black,linestyle=2):
  plot4:=plot(Pi/2,t=2*Pi..15,color=black,linestyle=2):
  plot5:=plot(-Pi/2,t=2*Pi..15,color=black,linestyle=2):
  display({plot1,plot2,plot3,plot4,plot5},title='resonance
    turned off');
```