Math 2250-1
Tuesday December 9
§9.3 Ecological models

- Finish predator-prey example Monday
  notice $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ trick for autonomous systems.
- then analyze the competition model:

$$\frac{dx}{dt} = \underbrace{a_1 x - b_1 x^2}_{\text{logistic}} \underbrace{- c_1 xy}_{\text{competition}}$$

all params > 0

$$\frac{dy}{dt} = \underbrace{a_2 y - b_2 y^2}_{\text{logistic}} - \underbrace{c_2 xy}_{\text{competition?}}$$

Equil. sol'ns:
$$x\left[a_1 - b_1 x - c_1 y\right] = 0$$
$$y\left[a_2 - b_2 y - c_2 x\right] = 0$$

$x = 0 \Rightarrow y(a_2 - b_2 y) = 0 \Rightarrow y = 0$ or $y = a_2/b_2$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ a_2/b_2 \end{bmatrix}$$

$x \neq 0 \Rightarrow a_1 - b_1 x - c_1 y = 0 \Rightarrow$ $y = 0$ or $a_2 - b_2 y - c_2 x = 0$

$y = 0 \Rightarrow x = \frac{a_1}{b_1}$

$$\begin{bmatrix} a_1/b_1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} b_1 & c_1 \\ c_2 & b_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

(book typo page 540 (8))

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{b_1 b_2 - c_1 c_2}\begin{bmatrix} b_2 & -c_1 \\ -c_2 & b_1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

call this $\begin{bmatrix} x_E \\ y_E \end{bmatrix}$

$$= \frac{1}{b_1 b_2 - c_1 c_2}\begin{bmatrix} a_1 b_2 - c_1 b_1 \\ -a_1 c_2 + b_1 a_2 \end{bmatrix}$$

[this may or may not be in 1st quadrant; depends on parameter values]

Text claims

1. If $c_1 c_2 < b_1 b_2$
   (relatively low competition between species vs. self-inhibitions)
   then $\begin{bmatrix} x_E \\ y_E \end{bmatrix}$ is asymptotically stable equil, and $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \to \begin{bmatrix} x_E \\ y_E \end{bmatrix}$

2. if $c_1 c_2 > b_1 b_2$, $\begin{bmatrix} x_E \\ y_E \end{bmatrix}$ is unstable, and either $x(t)$ or $y(t) \to 0$ as $t \to \infty$
   (with 100% probability)

(for any IVP if $x_0 > 0$, $y_0 > 0$, and if $\begin{bmatrix} x_E \\ y_E \end{bmatrix}$ is in 1st quad)

book shows examples - we'll work out the general proof!

Discussion of $\begin{bmatrix} x_E \\ y_E \end{bmatrix}$:

$$J = \begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix} = \begin{bmatrix} a_1 - 2b_1 x - c_1 y & -c_1 x \\ -c_2 y & a_2 - 2b_2 y - c_2 x \end{bmatrix}$$

at $\begin{bmatrix} x_E \\ y_E \end{bmatrix}$ we have $\begin{bmatrix} b_1 & c_1 \\ c_2 & b_2 \end{bmatrix} \begin{bmatrix} x_E \\ y_E \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$

so (trickery!) at this point

$$J = \begin{bmatrix} -b_1 x_E & -c_1 x_E \\ -c_2 y_E & -b_2 y_E \end{bmatrix}$$

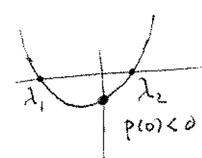
write $x = x_E$
$y = y_E$

$$p(\lambda) = |J - \lambda I| = (b_1 x + \lambda)(b_2 y + \lambda) - c_1 c_2 xy$$

$$p(\lambda) = \lambda^2 + (b_1 x + b_2 y)\lambda + (b_1 b_2 - c_1 c_2)\underbrace{xy}_{\substack{>0 \\ \text{by hypothesis}}}$$

the graph of
$p(\lambda)$ is a concave up parabola

(2) if $c_1 c_2 > b_1 b_2$ then $p(0) < 0$
so $p(\lambda)$ has a positive and a negative root, by intermediate value thm, so $\begin{bmatrix} x_E \\ y_E \end{bmatrix}$ is (unstable) saddle.

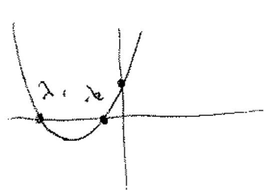


(1) if $c_1 c_2 < b_1 b_2$, then
vertex of parabola occurs at $\lambda = -\left(\frac{b_1 x + b_2 y}{2}\right)$ (complete square!)

and $p\left(-\frac{b_1 x + b_2 y}{2}\right) = \frac{1}{4}(b_1 x + b_2 y)^2 - \frac{1}{2}(b_1 x + b_2 y)^2 + (b_1 b_2 - c_1 c_2)xy$

$$= -\frac{1}{4}\left[\underbrace{b_1^2 x^2 + b_2^2 y^2 + 2b_1 b_2 xy - 4b_1 b_2 xy + 4c_1 c_2}_{(b_1 x - b_2 y)^2 + 4c_1 c_2}\right]$$

$$< 0$$



since $p(0) > 0$ in this case,
$p(\lambda)$ has two negative roots.

$\blacksquare$

(this proves the stability claims for the linearized problem, not the global claims for the nonlinear problem)

Example with numbers: (See picture page 5 if you want to cheat)

Example 3. p. 541

$$x' = 14x - \tfrac{1}{2}x^2 - xy \quad = x\left(14 - \tfrac{1}{2}x - y\right)$$
$$y' = 16y - \tfrac{1}{2}y^2 - xy \quad = y\left(16 - \tfrac{1}{2}y - x\right)$$

critical points

$$\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}0\\32\end{bmatrix}, \begin{bmatrix}28\\0\end{bmatrix}, \begin{bmatrix}12\\8\end{bmatrix}$$

We shall linearize at each critical point and then "guess" the phase portrait in the first quadrant.

$$J = \begin{bmatrix}F_x & F_y\\ G_x & G_y\end{bmatrix} = \begin{bmatrix}14 - x - y & -x\\ -y & 16 - y - x\end{bmatrix}$$

@ $\begin{bmatrix}0\\0\end{bmatrix}$ $\quad J = \begin{bmatrix}14 & 0\\ 0 & 16\end{bmatrix}$; nodal source $\quad \begin{bmatrix}u'\\v'\end{bmatrix} = \begin{bmatrix}14u\\16v\end{bmatrix}$

$\lambda = 14$ $\quad \lambda = 16$

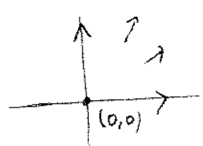
$\vec{v} = \begin{bmatrix}1\\0\end{bmatrix}$ $\quad \vec{v} = \begin{bmatrix}0\\1\end{bmatrix}$

$$\frac{dv}{du} = \frac{v'}{u'} = \frac{8}{7}\frac{v}{u}$$

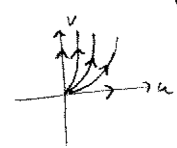
$$\frac{1}{v}\,dv = \frac{8}{7}\,\frac{1}{u}\,du \Rightarrow \ln|v| = \frac{8}{7}\ln|u| + C$$

$$v = Cu^{8/7}$$

rough:



more precise



@ $\begin{bmatrix}28\\0\end{bmatrix}$, $J = \begin{bmatrix}-14 & -28\\ 0 & -12\end{bmatrix}$

$\lambda = -12, -14$; nodal sink

$\vec{v} = \begin{bmatrix}14\\-1\end{bmatrix}$ $\quad \vec{v} = \begin{bmatrix}1\\0\end{bmatrix}$

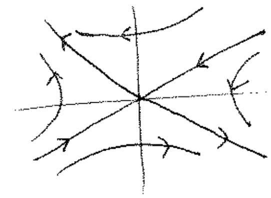
@ $\begin{bmatrix}0\\32\end{bmatrix}$, $J = \begin{bmatrix}-18 & 0\\ -32 & -16\end{bmatrix}$

$\lambda = -16, -18$ nodal sink

$\vec{v} = \begin{bmatrix}0\\1\end{bmatrix}$ $\quad \vec{v} = \begin{bmatrix}1\\16\end{bmatrix}$

a $\begin{bmatrix} 12 \\ 8 \end{bmatrix}$, $J = \begin{bmatrix} -6 & -12 \\ -8 & -4 \end{bmatrix}$

$$|J-\lambda I| = \begin{vmatrix} -6-\lambda & -12 \\ -8 & -4-\lambda \end{vmatrix}$$



```
> with(linalg):
Warning, the protected names norm and trace have been redefined and unprotected
> A:=matrix(2,2,[-6,-12,-8,-4]);
```

$$A := \begin{bmatrix} -6 & -12 \\ -8 & -4 \end{bmatrix}$$

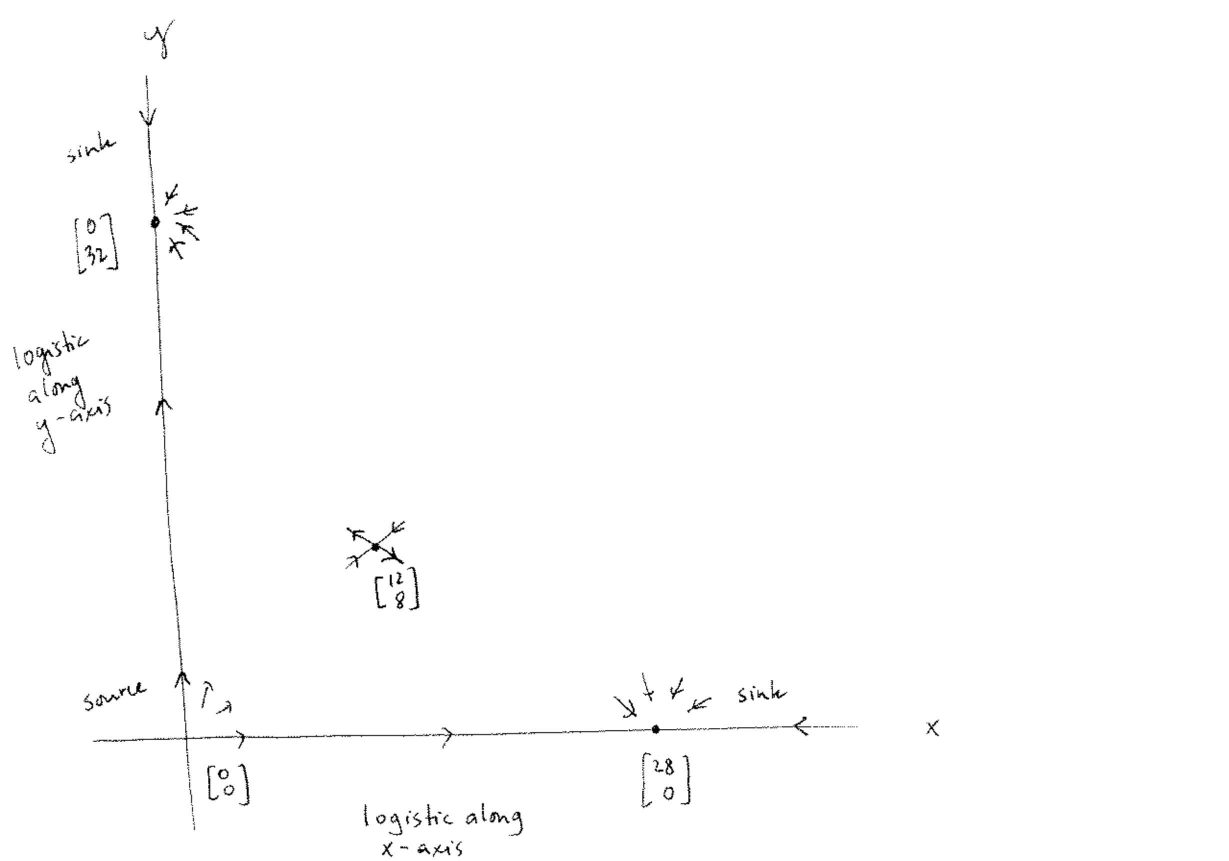
```
> eigenvectors(A);
```

$$\left[-5+\sqrt{97}, 1, \left\{\left[\frac{1}{8}-\frac{\sqrt{97}}{8}, 1\right]\right\}\right], \left[-5-\sqrt{97}, 1, \left\{\left[\frac{1}{8}+\frac{\sqrt{97}}{8}, 1\right]\right\}\right]$$

```
> evalf(eigenvectors(A));
```

[4.848857802, 1., {[-1.106107225, 1.]}], [-14.84885780, 1., {[1.356107225, 1.]}]

put it all together!! (then play with pplane)



along $x=100$
$0 \le y \le 100$

$x' \le 14(100) - 5000 < 0$

along $y = 100$
$0 \le x \le 100$

$y' \le 16(100) - 5000 < 0$

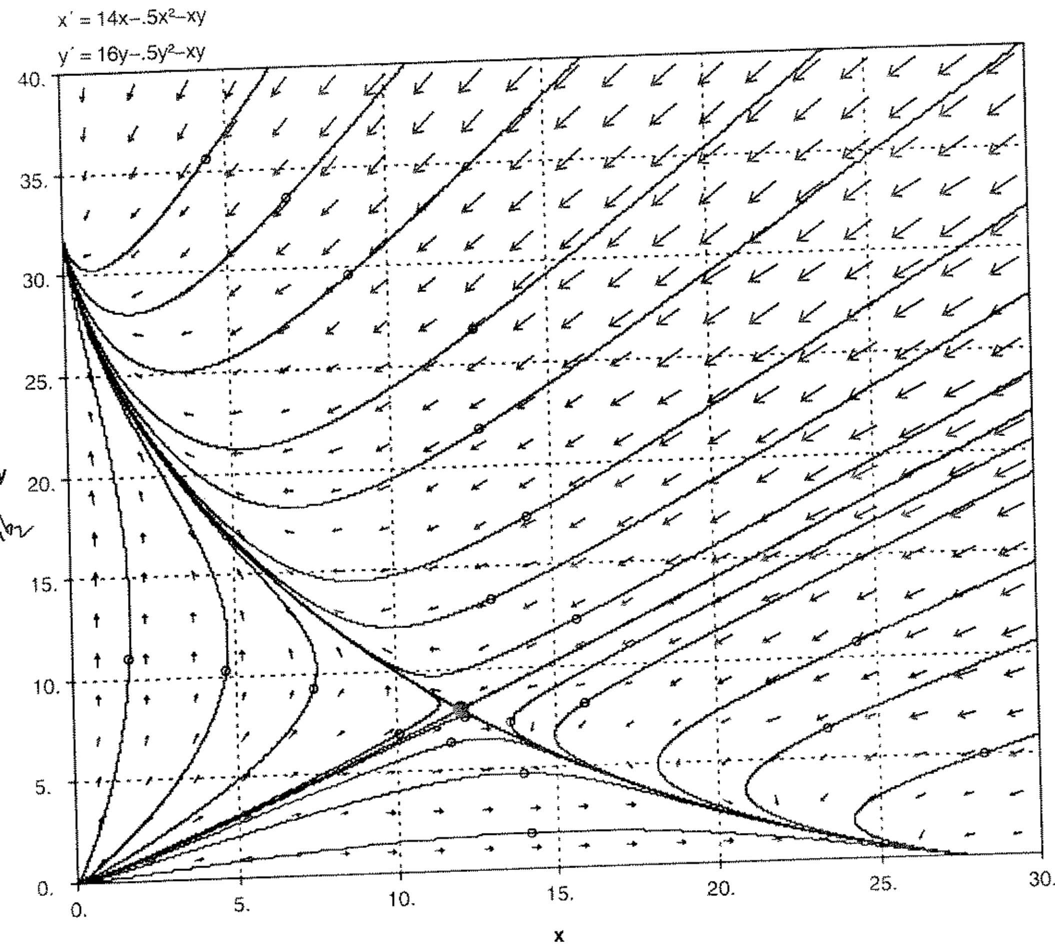
(huge populations decay)

Use thm: only possibilities for sol'n traj's $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ as $t \to \infty$ are

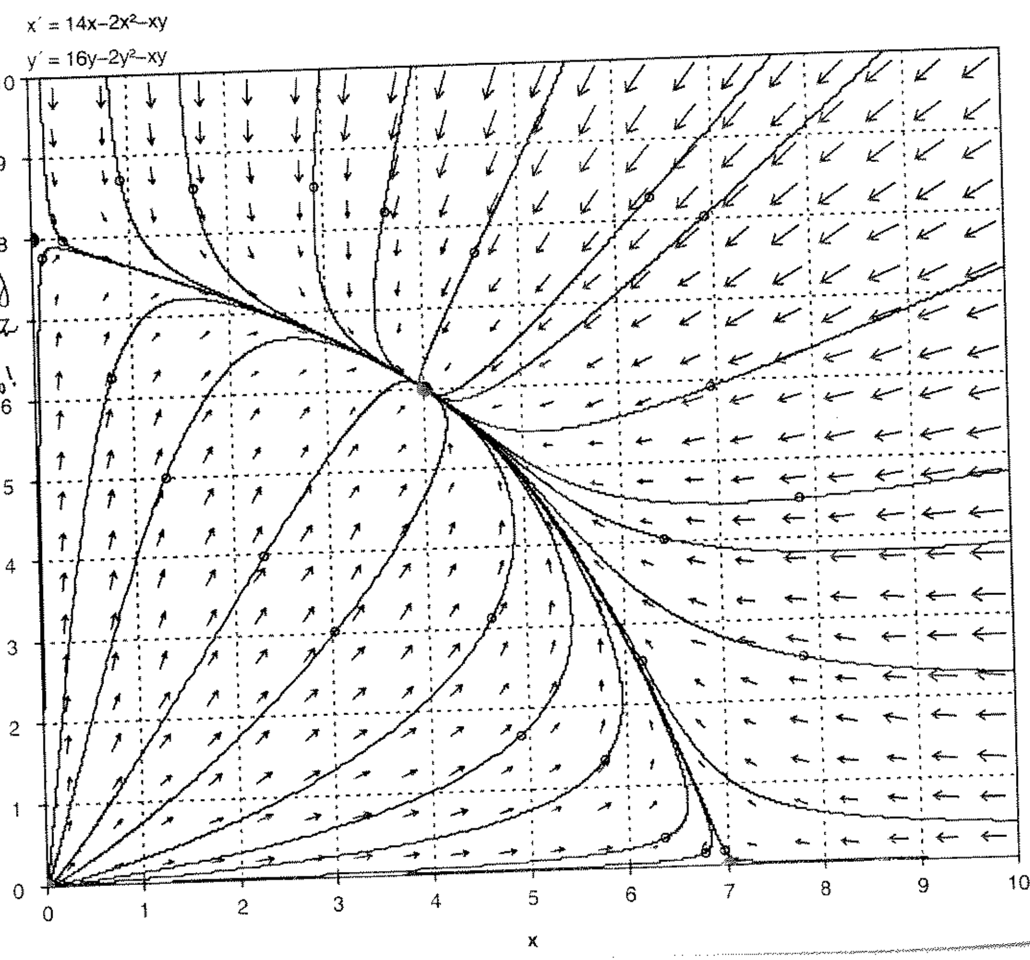
(1) $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \to \vec{x}_*$ equil

(2) $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \to \infty$

(3) $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \to$ periodic orbit



$c_1c_2 > b_1b_2$

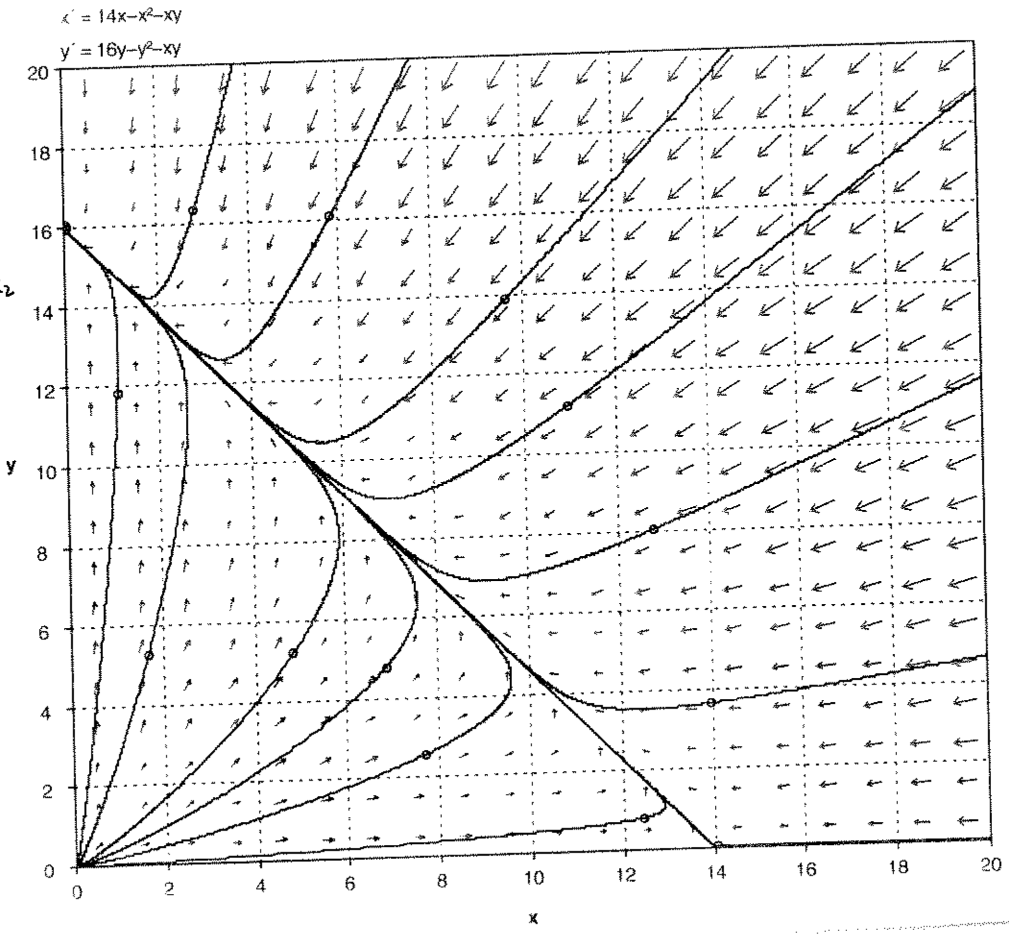


We studied exactly this system a few days ago!

$c_1c_2 < b_1b_2$

We didn't consider all possible cases, even of this "simple" competition model:

e.g. $b_1 b_2 = c_1 c_2$



e.g. $\begin{bmatrix} x_e \\ y_e \end{bmatrix}$ not in 1st quad.