Math 2250-1
Monday December 8

§9.2 Classification of equilibrium soltns via linearization
(& continue 1.3)

We've been studying the stability and geometry of equilibrium soltns to 2 1st order (non-linear) DE's, via linearization.

- Page 5 Friday example with complex eigendata

Here's how to name & classify these equilibria, for the linear systems, and the non-linear ones:

Eigenvalues of A	Type of Critical Point
Real, unequal, same sign	Improper node
Real, unequal, opposite sign	Saddle point
Real and equal	Proper or improper node
Complex conjugate	Spiral point
Pure imaginary	Center

FIGURE 9.2.9. Classification of the critical point (0, 0) of the two-dimensional system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

linear

Eigenvalues λ_1, λ_2 for the Linearized System	Type of Critical Point of the Almost Linear System
$\lambda_1 < \lambda_2 < 0$	Stable improper node
$\lambda_1 = \lambda_2 < 0$	Stable node or spiral point
$\lambda_1 < 0 < \lambda_2$	Unstable saddle point
$\lambda_1 = \lambda_2 > 0$	Unstable node or spiral point
$\lambda_1 > \lambda_2 > 0$	Unstable improper node
$\lambda_1, \lambda_2 = a \pm bi \quad (a < 0)$	Stable spiral point
$\lambda_1, \lambda_2 = a \pm bi \quad (a > 0)$	Unstable spiral point
$\lambda_1, \lambda_2 = \pm bi$	Stable or unstable, center or spiral point

FIGURE 9.2.12. Classification of critical points of an almost linear system.

nonlinear - dominated by linearization except in borderline cases.
(If $\lambda_1, \lambda_2 < 0$ or $\mathrm{Re}\,\lambda < 0$ for linearized problem, equil. pt is asympt. stable for non linear. If one $\lambda_i > 0$ or $\mathrm{Re}\,\lambda > 0$, then unstable. If $\lambda_1, \lambda_2 \le 0$ or $\mathrm{Re}\,\lambda = 0$, non-linear prob. is indeterminate).

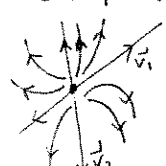
If λ real: $\vec{x}_H(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2$

(or if λ_1 is defective $\vec{x}_H(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_1 t}(t\vec{v}_1 + \vec{u})$)

$\lambda_1 < 0 < \lambda_2$

$\boxed{\text{saddle}}$ (always unstable)

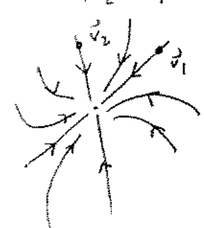
$0 < \lambda_1 < \lambda_2$



(unstable) $\boxed{\text{improper node}}$

$\lambda_1 = \lambda_2$ non-defective $\boxed{\text{proper node}}$

$\lambda_2 < \lambda_1 < 0$



asympt. (stable) $\boxed{\text{improper node}}$

$\lambda_1 = \lambda_2$ non-defect $\boxed{\text{proper}}$

$\lambda = p \pm qi$ complex

$\vec{v} = \vec{v}_1 \pm \vec{v}_2 i$

$$e^{(p+qi)t}(\vec{v}_1 + i\vec{v}_2)$$
$$= e^{pt}(\cos qt\,\vec{v}_1 - \sin qt\,\vec{v}_2) + i e^{pt}(\sin qt\,\vec{v}_1 + \cos qt\,\vec{v}_2)$$

so $\vec{x}_H(t) = c_1 e^{pt}(\cos qt\,\vec{v}_1 - \sin qt\,\vec{v}_2) + c_2 e^{pt}(\sin qt\,\vec{v}_1 + \cos qt\,\vec{v}_2)$

$p < 0$  asy. stable $\boxed{\text{spiral point}}$

$p > 0$  unstable $\boxed{\text{spiral pt}}$

$p = 0$  stable $\boxed{\text{center}}$ for linear problem, indeterminant for non-linear!

begin
§9.3 :

(An example where the linearization is borderline)

- predator-prey :

$$PP \quad \begin{cases} x'(t) = ax - pxy = x(a-py) & \text{prey} \\ y'(t) = -by + qxy = y(-b+qx) & \text{predator} \end{cases}$$

$a, b, p, q > 0$

natural region of interest : $x \geq 0, y \geq 0$

equil sol'ns:

$x = 0 \Downarrow y = 0$ $\quad \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$y = a/p \Downarrow x = \frac{b}{q}$ $\quad \begin{bmatrix} b/q \\ a/p \end{bmatrix}$ in 1st quad.
is this equil stable?

Linearize:

$$\begin{bmatrix} F \\ G \end{bmatrix} = \begin{bmatrix} ax - pxy \\ -by + qxy \end{bmatrix}$$

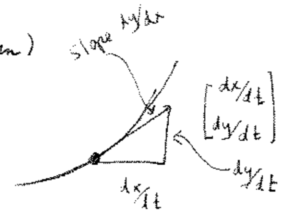
$$\text{Jacobian matrix} = \begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix} = \begin{bmatrix} a-py & -px \\ qy & -b+qx \end{bmatrix}$$

$$\text{at } \vec{x}_*, \quad J = \begin{bmatrix} 0 & -pb/q \\ aq/p & 0 \end{bmatrix}$$

$$|J - \lambda I| = \lambda^2 + ab \quad ; \quad \lambda = \pm i\sqrt{ab}$$

linearization DE is stable center (ellipse trajectories)
this is borderline for the original non-linear system....

method for understanding sol'n trajectories: (in this case for the non-linear problem)



1240 Chain rule $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$

So $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ (use this in §9.1 #24 HW too!)

for PP, $\frac{dy}{dx} = \frac{y(-b+qx)}{x(a-py)}$ separable!

$$\frac{a-py}{y}\,dy = \frac{-b+qx}{x}\,dx$$

$$a\ln y - py = -b\ln x + qx + C \quad \text{(in } 1^{st} \text{ quadrant!)}$$

$$\underbrace{a\ln y + b\ln x - py - qx}_{f(x,y)} = C$$

So solution trajectories follow level curves of $f(x,y)$ (See figures 6.3.1 & 6.3.2 page 401) and this $f(x,y)$ has a local max at $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b/q \\ a/p \end{bmatrix} = \vec{x}^*$ (Actually global 1^{st} quadrant max!)

So $\vec{x}^*$ is stable, and general sol'ns are periodic!!