Math 2250
Friday Dec. 9

Linearization: On Wednesday

we linearized the DE

$$\frac{dx}{dt} = 14x - 2x^2 - xy \qquad \text{rabbits}$$
$$\frac{dy}{dt} = \underbrace{16y - 2y^2}_{\text{logistic}} - \underbrace{xy}_{\text{competition}} \qquad \text{squirrels}$$

near the interesting equilibrium solution $\begin{bmatrix}4\\6\end{bmatrix}$

We wrote
$$x = 4 + u$$
$$y = 6 + v$$
so that small $\begin{bmatrix}u\\v\end{bmatrix}$ corresponds to $\begin{bmatrix}x\\y\end{bmatrix}$ close to $\begin{bmatrix}4\\6\end{bmatrix}$.

We got
$$\underbrace{\begin{bmatrix}u'\\v'\end{bmatrix} = \underset{\uparrow \text{ linear piece}}{\begin{bmatrix}-8 & -4\\-6 & -12\end{bmatrix}\begin{bmatrix}u\\v\end{bmatrix}}} + \underset{\uparrow \text{ error}}{\begin{bmatrix}-2u^2 - uv\\-2v^2 - uv\end{bmatrix}}$$

error; if $\left\|\begin{bmatrix}u\\v\end{bmatrix}\right\| < \underset{\text{tiny}}{\delta}$ then $\|\text{error}\| \le 6\underset{\text{tiny squared}}{\delta^2}$

this linear DE has

eigenvectors(A) = [-4.708.1, {[.77, -.63]}], [-15.3, 1 {[.49, .89]}

- Using the eigenvector data we found the general sol'n

$$\begin{bmatrix}u(t)\\v(t)\end{bmatrix} = c_1 e^{-4.7t}\begin{bmatrix}.77\\-.63\end{bmatrix} + c_2 e^{-15.3t}\begin{bmatrix}.49\\.89\end{bmatrix}$$ to the linearized problem

and so, for $|u|, |v|$ small,

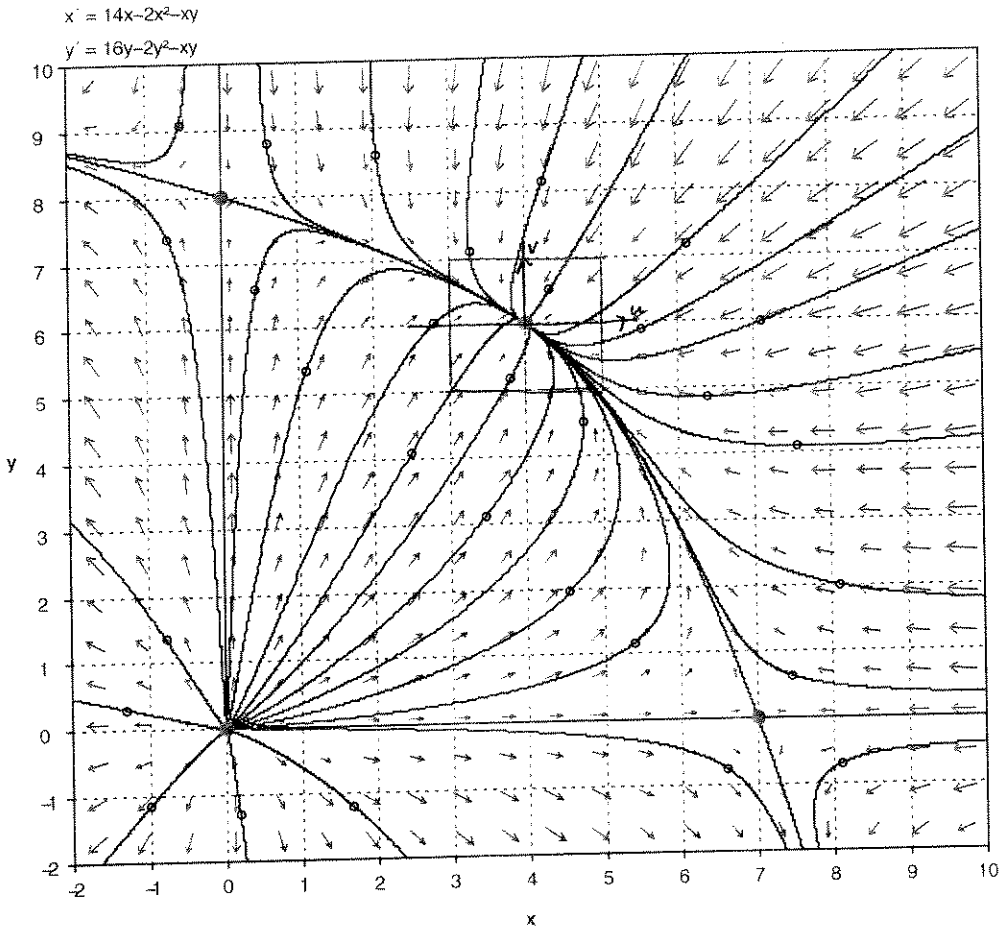
$$\begin{bmatrix}x(t)\\y(t)\end{bmatrix} \cong \begin{bmatrix}4\\6\end{bmatrix} + \begin{bmatrix}u(t)\\v(t)\end{bmatrix} \longrightarrow \begin{bmatrix}4\\6\end{bmatrix} \quad \text{as } t \to \infty$$

(An analysis theorem says that if the linearized problem has general sol'n that decays exponentially, then the equil. sol'n to the non-linear problem is (also) exponentially asymptotically stable.)

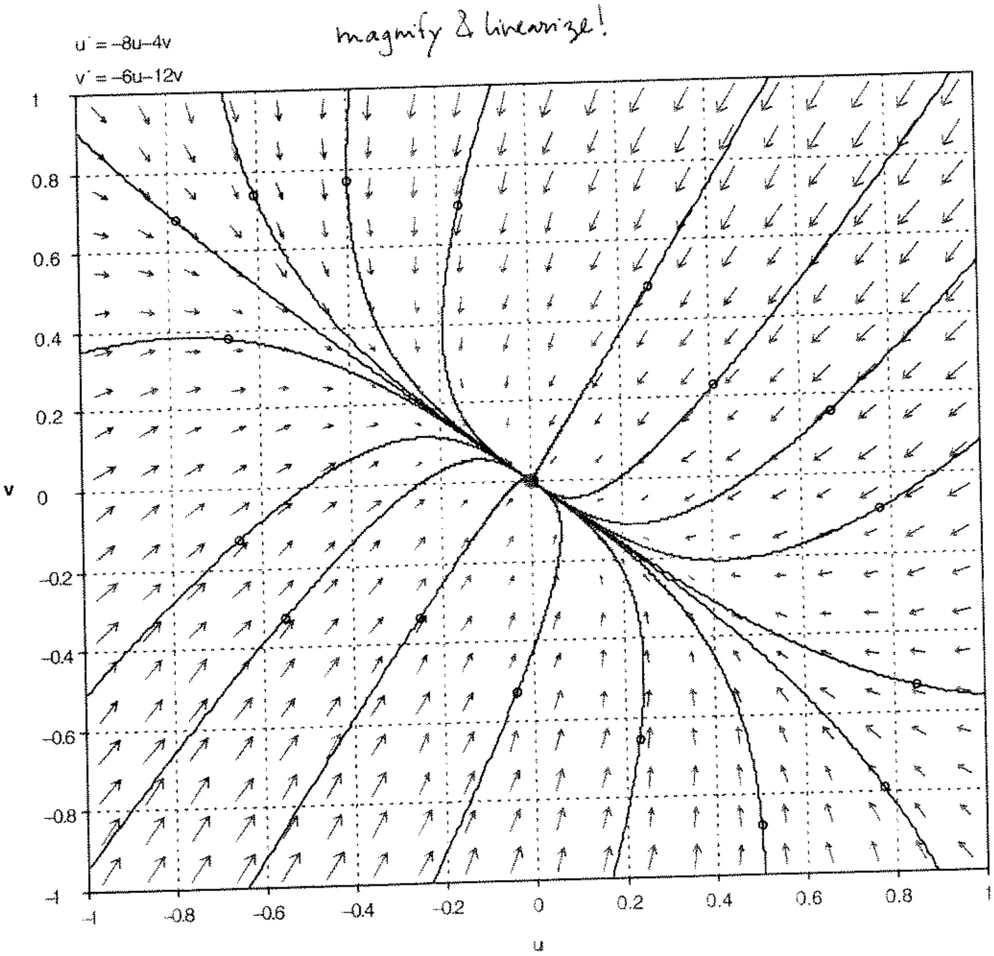
- We also understood, using the eigenvectors & evals from the matrix of the linearized problem, why the phase portrait for the linearized problem looks the way it does. (next page.)

④ Wed = ② Friday

What happens to rabbit-squirrel populations??



magnify & linearize!



Shortcut to linearization: (works for systems of n DE's; illustrated for $n=2$)

Let

$$(1)\quad \begin{cases} x' = F(x,y) \\ y' = G(x,y) \end{cases}$$

$$F(x_*,y_*) = F(P) = 0$$
$$G(x_*,y_*) = G(P) = 0$$

write
$$x(t) = x_* + u(t)$$
$$y(t) = y_* + v(t)$$

we are interested in what happens for $\|(u,v)\|$ small.

error; $\frac{\varepsilon}{\|(u,v)\|} \to 0$ as $(u,v)\to(0,0)$

$$x' = F(x_*+u,\, y_*+v) = \cancel{F(x_*,y_*)}^{\,0} + F_x(x_*,y_*)u + F_y(x_*,y_*)v + \varepsilon_1(u,v)$$
$$y' = G(x_*+u,\, y_*+v) = \cancel{G(x_*,y_*)}_{\,0} + G_x(-)u + G_y(-)v + \varepsilon_2(u,v)$$

Math 2210 affine approx.

$$u' = x' = F_x u + F_y v + \varepsilon_1(u,v)$$
$$v' = y' = G_x u + G_y v + \varepsilon_2(u,v)$$

where the partial derivs of F & G are evaluated at the equil. pt.

$$(2)\quad \begin{bmatrix} u' \\ v' \end{bmatrix} = \underbrace{\begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix}}_{\text{"A"}} \begin{bmatrix} u \\ v \end{bmatrix}$$

(Partial derivs in A are evaluated at (x^*,y^*))

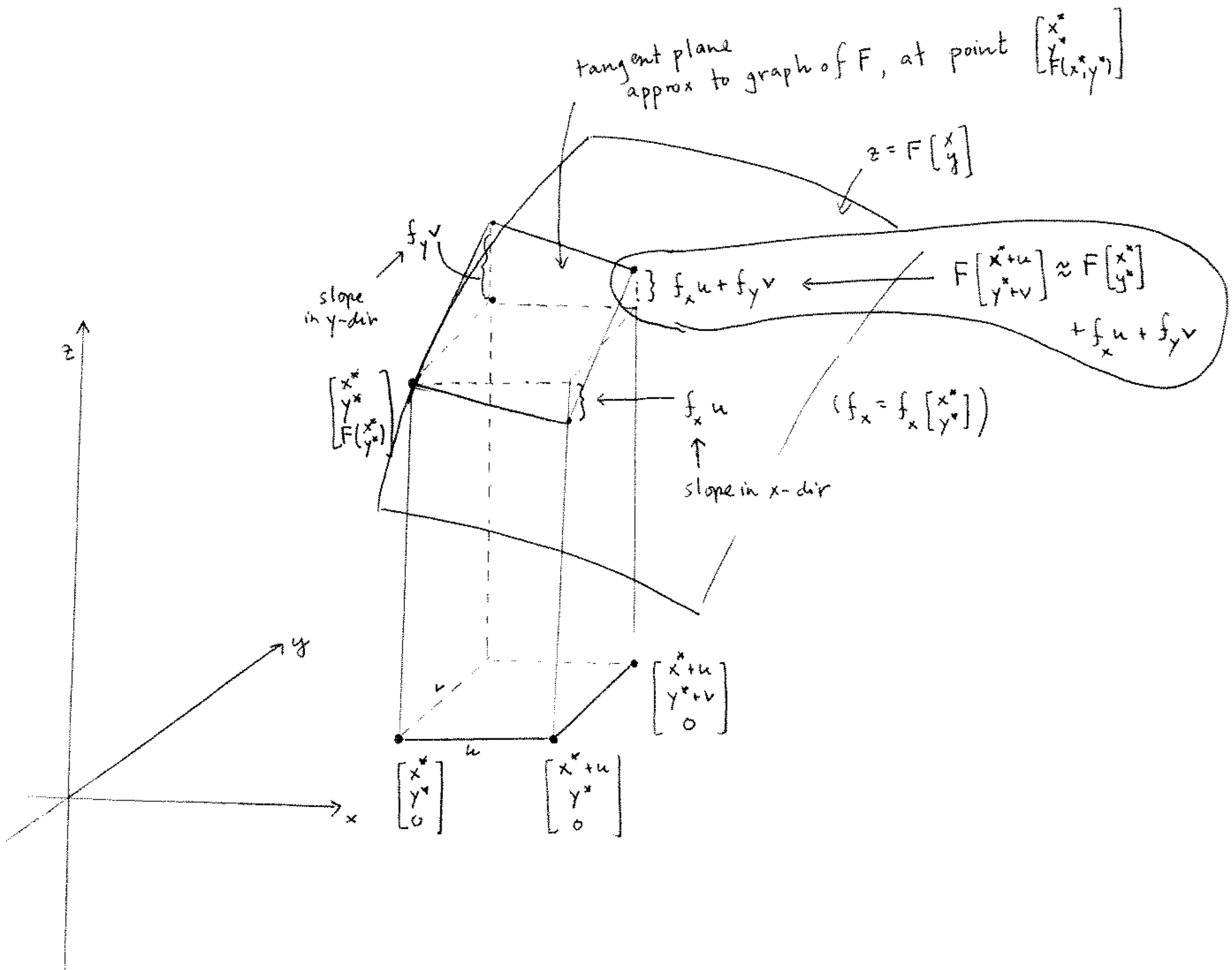
this is the linearization of (1), at (x^*,y^*). the eigenvector data of A determines stability for the nonlinear system (1), in the non borderline cases.

(We use the same letters u,v as we did for the non-linear problem, even though the sol'n $\begin{bmatrix} u \\ v \end{bmatrix}$ to the linearized problem only approximate the translated sol'n $\begin{bmatrix} u \\ v \end{bmatrix}$ to the non-linear problem)

the matrix A is called the <u>Jacobian matrix</u> for $\vec{F}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} F(x,y) \\ G(x,y) \end{bmatrix}$, at $\begin{bmatrix} x_* \\ y_* \end{bmatrix}$

This picture might help you recall the tangent plane approximation for a function of 2 variables.

(Math 2210 or 1260 or 1280)



<u>Exercise</u> Compute the linearizations of our rabbit-squirrel model on page 1 at the 4 equilibrium soltns we found Wednesday.

- Verify we get the same linear system at $\begin{bmatrix}4\\6\end{bmatrix}$ that we got "the long way"
- Carry out the same eigenvector analysis at $\begin{bmatrix}0\\0\end{bmatrix}$, $\begin{bmatrix}7\\0\end{bmatrix}$, $\begin{bmatrix}0\\8\end{bmatrix}$ (and linearization) to explain the behavior of the non-linear problem at those equilibria.

partial answer

$$J(x,y) = \begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix} = \begin{bmatrix} 14-2x-y & -x \\ -y & 16-4y-x \end{bmatrix}$$

We'll classify the equilibrium points of linear systems Monday, and it's a deep theorem that these classifications essentially carry over for equilibria in non-linear autonomous systems.

For today, find the equilibria for $x' = x - y - x^2 + xy$ (6.1 Hw # 8)
$y' = -y - x^2$
and linearize about the one in the 3rd quadrant
(complex eigenvectors!)

$$J = \begin{bmatrix} 1-2x+y & -1+x \\ -2x & -1 \end{bmatrix}$$