Math 2250-1
Wednesday Dec 3
§9.1-9.2
phase plane, equilibrium soltns, stability, linearization

HW for Friday Dec 12 (last assignment, hurray!!)
9.1 (5, 8, 11, 15, 20, 24)
9.2 5 6 7, 8, 9 14, 15, 19, 27, 30
on 19, 27 use pplane to find & classify the other equilibrium sol'ns besides (0,0), & print out the phase portrait
9.3 8 9, 10, 14, 15, 16, 17
on 8, 10 create a phase portrait for the nonlinear system (3) & explain how your linearization compares
9.4 12, 13, 14, 15, 16
the only technology output you need to hand in is what is requested here.

(nonlinear) system of two 1st order DE's

$$(1)\quad \begin{aligned} \frac{dx}{dt} &= F(x,y,t) \\ \frac{dy}{dt} &= G(x,y,t) \end{aligned}$$

example (§9.3)
$x(t)$ = prey population (fish, rabbits, etc.)
$y(t)$ = predator population (sharks, foxes, etc)

$$\begin{cases} \frac{dx}{dt} = ax - pxy \\ \frac{dy}{dt} = -by + qxy \end{cases}$$

($-cx^2$, if you want the prey to be logistic).

understand model assumptions:

example (§9.4)
$x(t)$ = pendulum angle $\theta(t)$
$y(t) = x'(t)$ = angular velocity $\theta'(t)$

$$\begin{cases} x' = y \\ y' = -\frac{g}{L}\sin x \end{cases}$$

Def: If the only dependence of F and G on t is through $x(t)$ & $y(t)$, then the system (1) is called autonomous, i.e.

$$(2)\quad \begin{aligned} \frac{dx}{dt} &= F(x,y) \\ \frac{dy}{dt} &= G(x,y) \end{aligned}$$

In this case we call x-y plane the phase plane, and the solution curves $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ are called trajectories.
They follow the tangent vector field $\begin{bmatrix} F \\ G \end{bmatrix}$

constant solutions to (2) are called equilibrium solutions. They are exactly the solutions to the (non)linear system

$$(3)\qquad \begin{aligned} 0 &= F(x,y) \\ 0 &= G(x,y) \end{aligned}$$

example: competing species; say
$x(t)$ = rabbit population
$y(t)$ = squirrel population

perhaps

$$\frac{dx}{dt} = 14x - 2x^2 - xy$$

$$\frac{dy}{dt} = \underbrace{16y - 2y^2}_{\text{logistic}} \underbrace{- xy}_{\text{competition}}$$

Find the equilibrium sol'ns

ans $(0,0)$
$(0,8)$
$(7,0)$
$(4,6)$

It will be important to know whether equilibrium sol'ns are stable or unstable

Def $\begin{bmatrix} x_* \\ y_* \end{bmatrix} = \vec{x}^*$ is a stable equilibrium for (2) if it is a constant sol'n (equilibrium) (i.e. satisfies (3)), and if $\forall \varepsilon > 0 \ \exists \ \delta > 0$ s.t.
whenever $\|\vec{x}^* - \vec{x}_0\| < \delta$ $\quad \left(\|\vec{x}^* - \vec{x}_0\| = \sqrt{(x_0 - x^*)^2 + (y_0 - y^*)^2} \right)$
then the sol'n to (2) with $\vec{x}(0) = \vec{x}_0$ satisfies $\|\vec{x}(t) - \vec{x}^*\| < \varepsilon \quad \forall \ t > 0.$

$\begin{bmatrix} x_* \\ y_* \end{bmatrix}$ is unstable equilibrium if it is an equilibrium sol'n which is not stable.

$\begin{bmatrix} x_* \\ y_* \end{bmatrix}$ is asymptotically stable iff it is stable and $\exists \ \delta > 0$ s.t.
$\|\vec{x}^* - \vec{x}_0\| < \delta \implies$ the IVP sol'n with $\vec{x}_0 = \vec{x}(0)$ satisfies $\lim_{t \to \infty} \vec{x}(t) = \vec{x}^*$

We will understand stability by <u>linearizing</u> near equilibrium sol'tns.

<u>Example</u> : linearize rabbit-squirrel model near $\begin{bmatrix}4\\6\end{bmatrix}$

Let $x = 4 + u$ with $|u|, |v|$ small
$y = 6 + v$

Then $\frac{du}{dt} = \frac{dx}{dt} = 14(4+u) - 2\underbrace{(4+u)^2}_{(16+8u+u^2)} - (4+u)(6+v) =$

$= \cancel{56} \; -\cancel{32} \; -\cancel{24} \quad + u(14-16-6) \quad + v(-4) \quad + -4u^2 - uv$

$\frac{dv}{dt} = \frac{dy}{dt} = 16(6+v) - 2\underbrace{(6+v)^2}_{(36+12v+v^2)} - (4+u)(6+v)$

$= \cancel{96} \; -\cancel{72} \; -\cancel{24} \quad + u(-6) \quad + v(16-24-4) \quad -2v^2 - uv$

$$\begin{bmatrix}\frac{du}{dt}\\ \frac{dv}{dt}\end{bmatrix} = \underbrace{\begin{bmatrix}-8 & -4\\ -6 & -12\end{bmatrix}\begin{bmatrix}u\\ v\end{bmatrix}}_{\text{linear piece}} + \underbrace{\begin{bmatrix}-4u^2-uv\\ -2v^2-uv\end{bmatrix}}_{\text{error}}$$

error ; if $\left\|\begin{bmatrix}u\\v\end{bmatrix}\right\| < \delta$
then $\|\text{error}\| \le \delta^2 (8)$ is tiny.

linearization $\begin{bmatrix}u'\\ v'\end{bmatrix} = \begin{bmatrix}-8 & -4\\ -6 & -12\end{bmatrix}\begin{bmatrix}u\\ v\end{bmatrix}$

Use Maple output (which you are free to do in this chapter!!) to write down the general soltn

```
> with(linalg):
> Digits:=5:
> A:=matrix(2,2,[-8,-4.0,-6,-12]);
  eigenvectors(A);
```

$$A := \begin{bmatrix}-8 & -4.0\\ -6 & -12\end{bmatrix}$$

[-15.292, 1, {[0.48950, 0.89224]}], [-4.7085, 1, {[0.77217, -0.63541]}]

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>
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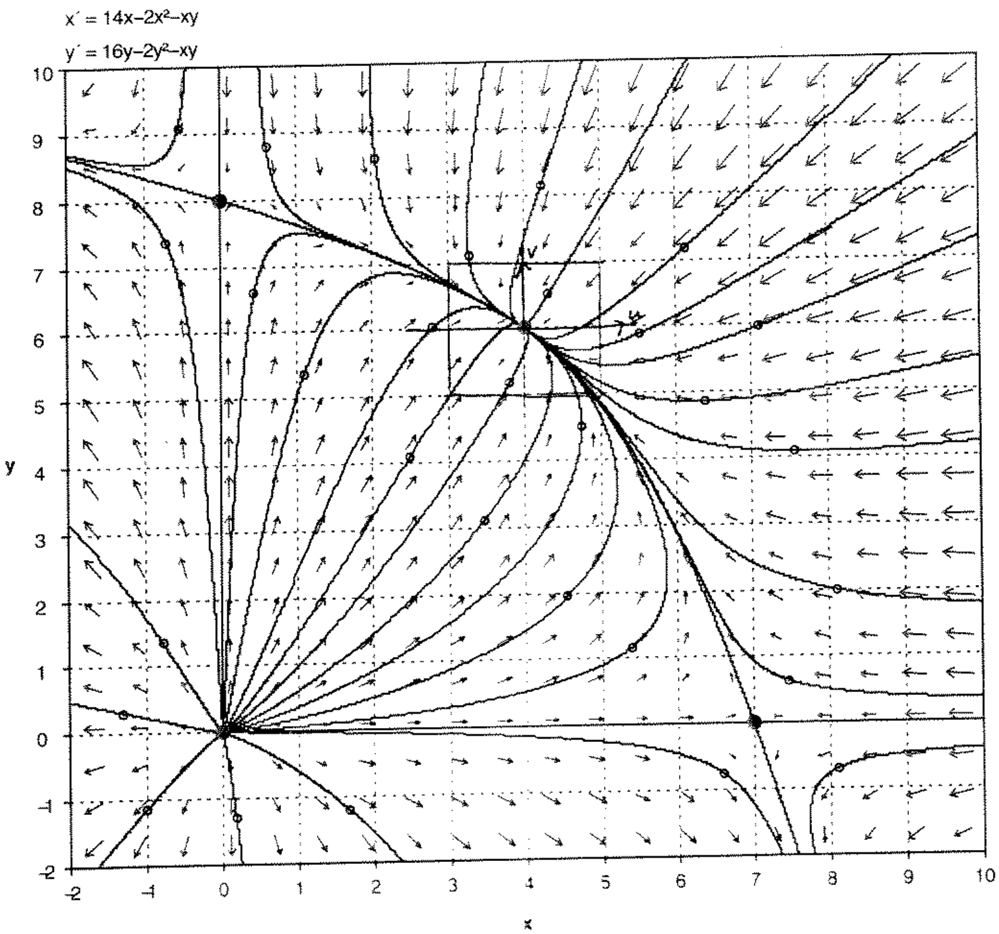
$$\begin{bmatrix}u(t)\\ v(t)\end{bmatrix} =$$

Then understand why the solution trajectories for $\begin{bmatrix}u\\v\end{bmatrix}$ look like they do on the next page (bottom) and what this has to do with the nonlinear phase portrait (top)!



What happens to rabbit-squirrel populations??

magnify & linearize!