Math 2250-1

Tuesday December 2 : First finish Monday's notes!

§7.5 Defective eigenspaces.

and how to find solution space to

$$\frac{d\vec{x}}{dt} = A\vec{x}$$

when $A$ is <u>not</u> diagonalizable.

<u>Example</u> Find the solution space to

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} 3 & 4 \\ 0 & 3 \end{bmatrix}\vec{x}$$

<u>step 1</u> (as always).

$$\begin{vmatrix} 3-\lambda & 4 \\ 0 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 = (\lambda-3)^2$$

$\lambda = 3$:

$$\begin{array}{cc|c} 0 & 4 & 0 \\ 0 & 0 & 0 \end{array} \qquad \text{eigenbasis } \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$$

$v_2 = 0$

$\vec{v} = t\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ ; <u>one</u> sol'n $e^{3t}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ ; need another independent sol'n!

<u>discussion</u> : let $A\vec{v} = \lambda_i\vec{v}$, but the $\lambda_i$-eigenspace is "defective" i.e. $\dim(\lambda_i\text{-eigenspace}) < k_i$ where charact poly has factor $(\lambda-\lambda_i)^{k_i}$.

Does multiplying by $t$ work?

i.e. we know (for $\lambda = \lambda_i$)

$$\vec{x}(t) = e^{\lambda t}\vec{v} \quad \text{solves} \quad \vec{x}' = A\vec{x}$$

does $\vec{z}(t) = te^{\lambda t}\vec{v}$ :

$$\vec{z}' = \left(e^{\lambda t} + t\lambda e^{\lambda t}\right)\vec{v} \qquad A\vec{z} = te^{\lambda t}A\vec{v} = t\lambda e^{\lambda t}\vec{v}$$

$\neq$

failure.

well, how about

$$\vec{z}(t) = e^{\lambda t}(t\vec{v} + \vec{w}) \quad (\text{same } \vec{v})$$

$$\vec{z}'(t) = \left[e^{\lambda t} + \cancel{t\lambda e^{\lambda t}}\right]\vec{v} + \lambda e^{\lambda t}\vec{w}$$

$$A\vec{z} = t\lambda e^{\lambda t}\vec{v} + e^{\lambda t}A\vec{w}$$

Success if

$$A\vec{w} = \lambda\vec{w} + \vec{v}$$

"a chain of length 2"

$$\boxed{\begin{aligned} (A-\lambda I)\vec{w} &= \vec{v} \\ (A-\lambda I)\vec{v} &= 0 \end{aligned}}$$

see next page also

<u>in our example</u>

$(A-\lambda I)\vec{w} = \vec{v}$ :

$$\begin{array}{cc|c} 0 & 4 & 1 \\ 0 & 0 & 0 \end{array}$$

$4w_2 = 1$ e.g. $\vec{w} = \begin{bmatrix} 0 \\ 1/4 \end{bmatrix}$

$$\vec{z}(t) = e^{3t}\left(t\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/4 \end{bmatrix}\right)$$

General sol'n:

$$\vec{x}_H(t) = c_1 e^{3t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{3t}\left(t\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/4 \end{bmatrix}\right)$$

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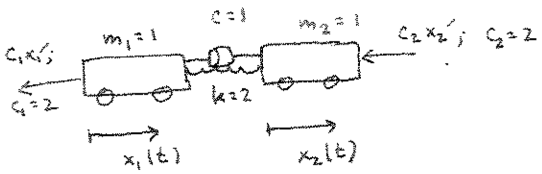


$$m_1 x_1'' = +k(x_2 - x_1) - c_1 x_1' + c(x_2' - x_1')$$
$$m_2 x_2'' = -k(x_2 - x_1) - c_2 x_2' - c(x_2' - x_1')$$

$$\begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} x_1' \\ x_2' \end{bmatrix}$$

cannot use §5.3 (no damping.) ← undamped sol'n is interesting too!
Must convert to 1st order system

$x_1 = x_1$
$x_2 = x_2$
$x_3 = x_1'$
$x_4 = x_2'$

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 2 & -3 & 1 \\ 2 & -2 & 1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Maple to the rescue! (See next page).

$$\begin{bmatrix} x_1 \\ x_2 \\ x_1' \\ x_2' \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + e^{-2t}\left[ c_2 \begin{bmatrix} 1 \\ 1 \\ -2 \\ -2 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ -1 \\ -2 \\ 2 \end{bmatrix} + c_4 \overbrace{\left[ t \begin{bmatrix} -1 \\ 1 \\ 2 \\ -2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right]}^{t\vec{u}_1 + \vec{u}_2} \right]$$

explain.

nice physically; the sum of MAPLE's 2 $\lambda = -2$ eigenvects

the difference of Maple's 2 $\lambda = -2$ eigenvects. also nice physical interpretation

```
> with(linalg):
Warning, the protected names norm and trace have been redefined and unprotected
> A:=matrix(4,4,[0,0,1,0,0,0,0,1,-2,2,-3,1,2,-2,1,-3]);
```

$$A := \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 2 & -3 & 1 \\ 2 & -2 & 1 & -3 \end{bmatrix}$$

```
> eigenvects(A);
```

$$[0, 1, \{[1, 1, 0, 0]\}], [-2, 3, \{[1, 0, -2, 0], [0, 1, 0, -2]\}]$$

the lambda = -2 eigenvalue is defective!!!!

```
> Iden:=array(1..4,1..4,identity);
```

$$Iden := \mathrm{array}(identity, 1 .. 4, 1 .. 4, [\,])$$

```
> evalm(Iden);
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

```
> B:=evalm(A+2*Iden); #the matrix A-lambda*I
```

$$B := \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ -2 & 2 & -1 & 1 \\ 2 & -2 & 1 & -1 \end{bmatrix}$$

```
> kernel(B^3); #this is the generalized lambda=-2
    #eigenspace.  It is indeed 3-dimensional
```

$$\{[1, 0, -2, 0], [0, 1, -2, 0], [0, 0, -1, 1]\}$$

```
> kernel(B^2);  #I will need a chain of length 2,
    #there won't be a chain of length 3 since
    #the actual eigenspace is 2-d
```

$$\{[1, 0, -2, 0], [0, 1, -2, 0], [0, 0, -1, 1]\}$$

```
> u2:=([0, 0, -1, 1]); #for this chain don't start
     #with an actual eigenvector.  I like this
     #vector because of its physical meaning in
     #the train problem!
```

$$u2 := [0, 0, -1, 1]$$

```
> u1:=evalm(B&*u2);
```

$$u1 := [-1, 1, 2, -2]$$

```
> evalm(B&*u1);  #should be zero
```

$$[0, 0, 0, 0]$$