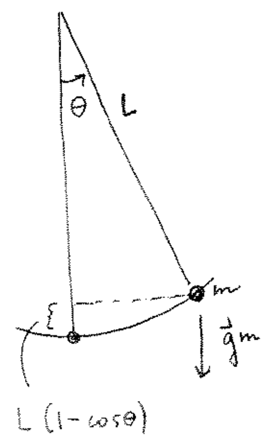
Math 2250-1
Wednesday Dec. 10
§9.4 Phase portraits for mechanical systems

Final exam
Tuesday December 16 8-10 a.m.
here (ST 104)...
Practice exam will be posted.
Friday will be a review of entire course
Saturday review session (practice exam) 11-1.
room to be announced.

rigid-rod pendulum:



(1) $\theta''(t) + \frac{g}{L}\sin\theta = 0$

We derived this DE using

$KE + PE \equiv \text{constant}$

$\frac{1}{2}mL^2(\theta')^2 + mgL(1-\cos\theta) \equiv \text{const}$

(2) $\frac{1}{2}L(\theta')^2 + g(1-\cos\theta) \equiv \widetilde{\text{const}}$

(and we took $\frac{d}{dt}(\%)$ to get (1))

For
$x = \theta(t)$
$y = \theta'(t)$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} y \\ -\frac{g}{L}\sin x \end{bmatrix}$$

equil sol'ns $y = 0$
$\sin x = 0$: $x = k\pi$, $k \in \mathbb{Z}$ $(0, \pm 1, \pm 2, \ldots)$

Linearization:

$$J = \begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L}\cos x & 0 \end{bmatrix}$$

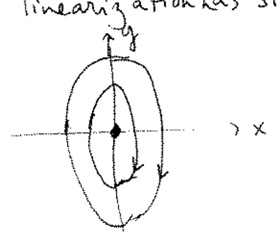
if $x = k\pi$ k even

$$J = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L} & 0 \end{bmatrix}$$

$|J - \lambda I| = \lambda^2 + \left(\frac{g}{L}\right) = 0$

$\lambda = \pm i\sqrt{\frac{g}{L}}$

linearization has stable center



(why clockwise?)

if ~~...~~ $x = k\pi$, k odd

$$J = \begin{bmatrix} 0 & 1 \\ \frac{g}{L} & 0 \end{bmatrix}$$

$|J - \lambda I| = \lambda^2 - \frac{g}{L} = 0$

$\lambda = \pm\sqrt{\frac{g}{L}}$ saddle

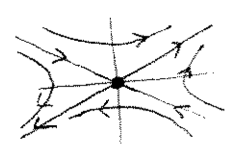
$\lambda = \sqrt{\frac{g}{L}}$

$$\left[\begin{array}{cc|c} -\sqrt{\frac{g}{L}} & 1 & 0 \\ \frac{g}{L} & -\sqrt{\frac{g}{L}} & 0 \end{array}\right]$$

$\vec{v} = \begin{bmatrix} 1 \\ \sqrt{\frac{g}{L}} \end{bmatrix}$

$\lambda = -\sqrt{\frac{g}{L}}$

$\vec{v} = \begin{bmatrix} -1 \\ \sqrt{\frac{g}{L}} \end{bmatrix}$



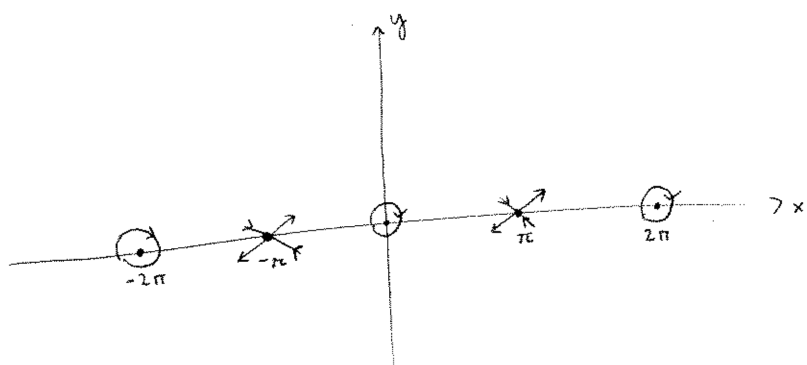
fill in ?

Hint : the trajectories must follow the level curves of the (scaled) total energy function (2), on page 1, i.e.

$$\frac{1}{2} L y^2 + g(1-\cos x) = \text{const} \longleftarrow \text{RHS is } 2\pi\text{-periodic in } x.$$

notice this function of (x,y) attains its minimum value of zero, at all $(x,y) = (k\pi, 0)$ with k even) (so $\cos x = 1$

and the Hessian matrix of this function is diagonal with positive diagonal entries, at these points, so the nearby trajectories for our system are (nearly) ellipses, for the non-linear system, so $(k\pi, 0)$ is stable center for the nonlinear problem, when k is even!



Add damping:

$$\Theta'' + c\Theta' + \frac{g}{L}\sin(\theta) = 0$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} y \\ -\frac{g}{L}\sin x - cy \end{bmatrix}$$

same equilibria (iff $y=0$ & $\sin x = 0$)

$$J = \begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L}\cos x & -c \end{bmatrix}$$

$$|J - \lambda I| = -\lambda(-\lambda - c) + \frac{g}{L}\cos x$$

$$= \lambda^2 + c\lambda \pm \frac{g}{L} \qquad \begin{pmatrix} +\frac{g}{L} \text{ if } x = k\pi,\ k \text{ even} \\ -\frac{g}{L} \text{ if } k \text{ odd} \end{pmatrix}$$

roots $\lambda = \dfrac{-c \pm \sqrt{c^2 \mp \frac{4g}{L}}}{2}$

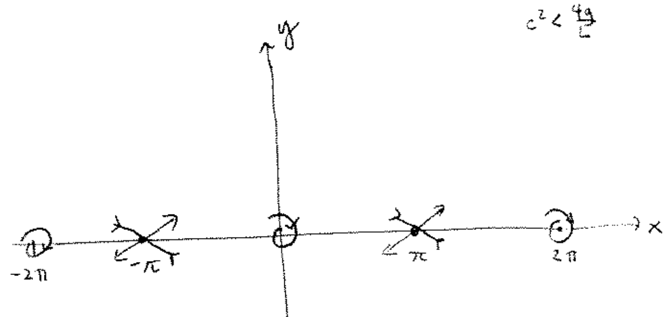
k odd $\Rightarrow$ saddle

k even, $c^2 < \frac{4g}{L} \Rightarrow$ stable spiral (underdamped)

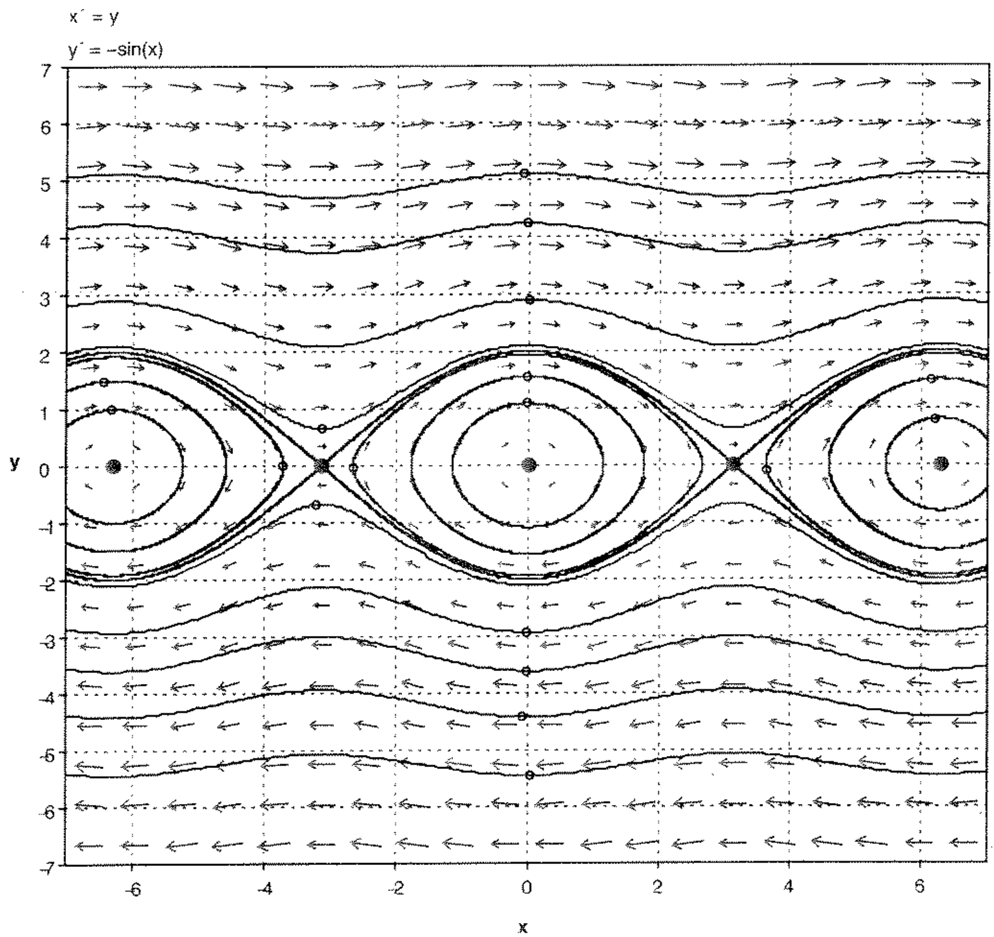
$c^2 > \frac{4g}{L} \Rightarrow$ stable node (overdamped)

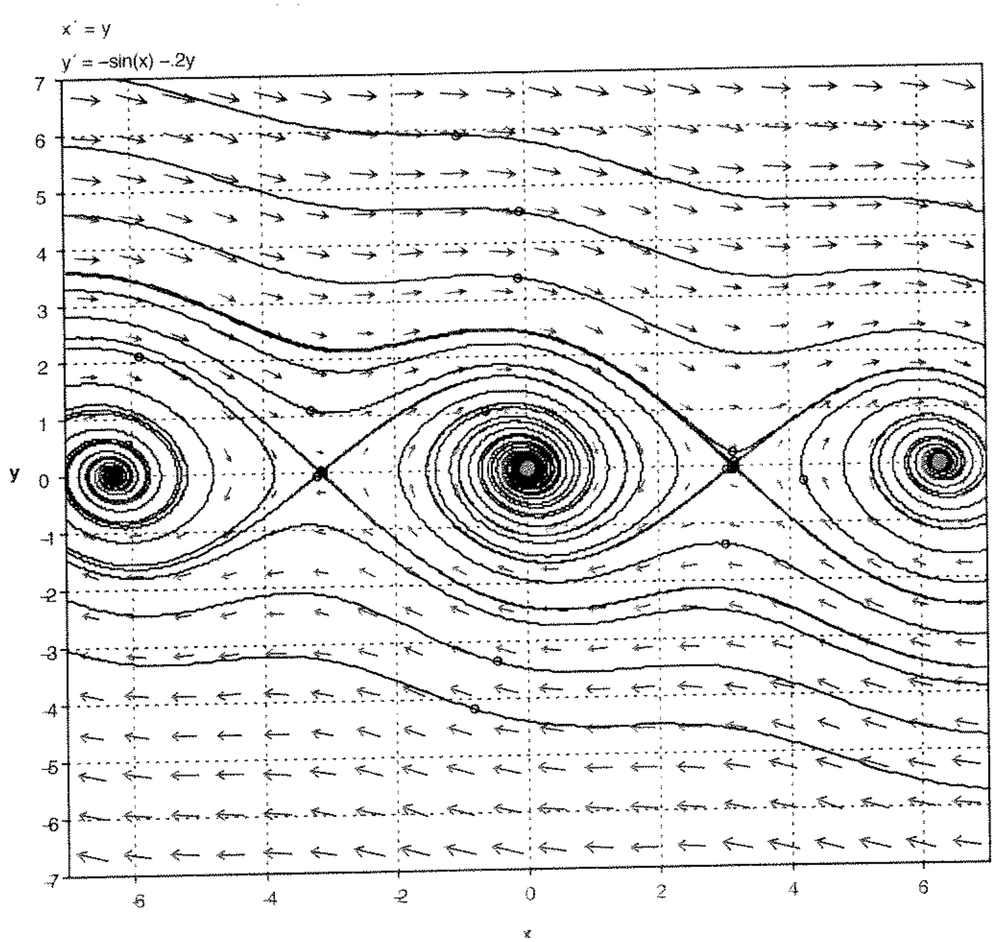
Fill in?

$c^2 < \frac{4g}{L}$



pictures for page 1-2:





nonlinear springs

$$m x'' = -c x' - k x + \beta x^3$$ (spring force is odd fcn; $F(-x) = -F(x)$)

$\beta > 0$ "soft spring"

$\beta < 0$ "hard spring"

Example 1 p. 553

$$x'' + 4x - x^3 = 0 \qquad \begin{matrix} c=0 \\ (k=4,\ \beta=1) \end{matrix}$$

has a E fcn which is constant along trajectories:

$$E = \tfrac{1}{2}(x')^2 + 2x^2 - \frac{x^4}{4} \equiv \text{const}$$!!

$x = x$, $y = x'$ $\Rightarrow$ $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} y \\ x^3 - 4x \end{bmatrix}$ $\longrightarrow$ $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{x^3-4x}{y}$ so $y\,dy = (x^3 - 4x)\,dx$

$$\tfrac{1}{2}y^2 = \frac{x^4}{4} - 2x^2 + C$$

equil: $\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \end{bmatrix}$

you can do the linearization analysis & deduce the phase portrait page 553, and below

Would such a spring make sense physically, at least for large x?