Math 2250-1
Monday December 1

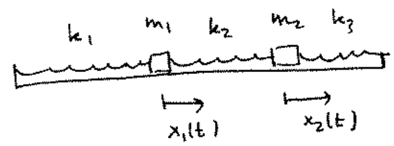
$\boxed{\text{Your earthquake project is posted! Due Monday Dec. 8.}}$

§ 7.4b : forced undamped mass-spring systems

Last lecture we discussed unforced undamped spring systems:

$$M\vec{x}'' = K\vec{x}$$

e.g. (done Wed)



$$m_1 x_1'' = k_2(x_2 - x_1) - k_1 x_1$$
$$m_2 x_2'' = -k_3 x_2 - k_2(x_2 - x_1)$$

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix} = \begin{bmatrix} -k_1-k_2 & k_2 \\ k_2 & -k_2-k_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$M\vec{x}'' = K\vec{x}$$

which (multiplying by M^{-1} on the left) yields

$$\vec{x}'' = A\vec{x} \qquad (A = M^{-1}K)$$

"A" for acceleration!

e.g. $$\begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix} = \begin{bmatrix} \frac{-k_1-k_2}{m_1} & \frac{k_2}{m_1} \\ \frac{k_2}{m_2} & \frac{-k_2-k_3}{m_2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Today let's force the same mass-spring systems!
(and study practical resonance for related slightly damped systems).

$$M\vec{x}'' = K\vec{x} + \vec{F}(t)$$

$$\boxed{* \qquad \vec{x}'' = A\vec{x} + \vec{f}(t)}$$

$A = M^{-1}K$
$\vec{f} = M^{-1}\vec{F}$

we study the case $\boxed{\vec{f}(t) = \vec{F}_0 \cos\omega t}$, so we're forcing each mass with the same angular frequency ω, but with different amplitudes for each mass.

constant vector of ± forcing amplitudes on each mass

(In practice all of the amplitudes except one may be zero)

General soln to * (linear nonhomogeneous sys of DE's) is

$$\vec{x} = \vec{x}_p + \vec{x}_H$$

discussed Wed: basis solns of form $\cos\omega_i t\,\vec{v}$, $\sin\omega_i t\,\vec{v}$ with $A\vec{v} = -\omega_i^2\vec{v}$.

try $\vec{x}_p = \vec{c}\cos\omega t$ (method of undetermined coeff's)

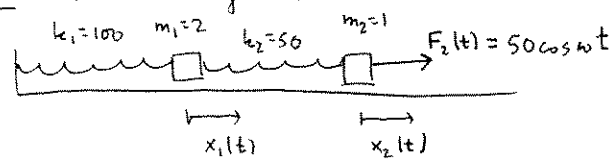
$*$ says $-\omega^2 \cancel{\cos\omega t}\,\vec{c} = \cancel{\cos\omega t}\,A\vec{c} + \cancel{\cos\omega t}\,\vec{F}_0$

$$-\vec{F}_0 = A\vec{c} + \omega^2\vec{c} = (A + \omega^2 I)\vec{c}.$$

$$\boxed{-\vec{F}_0 = (A+\omega^2 I)\vec{c}} \Longleftrightarrow \boxed{\vec{c} = -(A+\omega^2 I)^{-1}\vec{F}_0}$$

works for $\omega \neq \omega_i$, one of the natural frequencies

<u>Example</u> For the system



A) Using Newton's law (i.e. don't just plug into page 1) derive the 2^{nd} order system of DE's that governs this system, in the form of * on page 1

<u>Ans</u>

$$\begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix} = \begin{bmatrix} -75 & 25 \\ 50 & -50 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \cos\omega t \begin{bmatrix} 0 \\ 50 \end{bmatrix}$$

B) Find $\vec{x}_H$ for this system, using the algorithm we discussed on Wed.
(Example 1 page 431)

<u>Ans</u>

$$\vec{x}(t) = \underbrace{\underbrace{(c_1\cos 5t + c_2\sin 5t)}_{C_1\cos(5t-\alpha)} \begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{\text{(slower) in phase fundamental mode}} + \underbrace{\underbrace{(c_3\cos 10t + c_4\sin 10t)}_{C_2\cos(10t-\alpha)} \begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{\text{(faster) opposite phase fund. mode.}}$$

C) Find $\vec{x}_p = \cos\omega t\, \vec{c}$ to the inhomog. system

$$\begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix} = \begin{bmatrix} -75 & 25 \\ 50 & -50 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \cos\omega t \begin{bmatrix} 0 \\ 50 \end{bmatrix}$$

(Example 3 p. 437)
(Adjoint inverse formula or Cramer's rule will help here.)

ans
$$\vec{x}_p(t) = \cos\omega t \begin{bmatrix} \dfrac{1250}{(\omega^2-25)(\omega^2-100)} \\[2ex] \dfrac{50(\omega^2-75)}{(\omega^2-25)(\omega^2-100)} \end{bmatrix}$$

D) Interpretation: Notice our $\underbrace{\vec{x}_p \text{ doesn't work for } \omega=5 \text{ or } \omega=10.}_{\text{resonance}}$ (because we'd get resonance instead!)

beating

Even when $\omega \neq 5, 10$, if ω is near 5 or 10 the general soln

$$\vec{x}(t) = \vec{x}_p + \vec{x}_H$$

$\uparrow$ ang freq ω $\nwarrow$ superposition with angular frequencies 5, 10

is likely to exhibit large amplitude beating behavior because of the interaction between ω and its nearby natural angular frequency.

practical resonance

In any "real" demonstration of this model there will be some damping. Then, for the model which takes damping into account,

$$\vec{x}(t) = \vec{x}_p(t) + \vec{x}_H(t)$$

$\vec{x}_p(t) \to \cos\omega t\, \vec{c}(\omega) = \vec{x}_{sp}(t)$ (steady periodic)

$\vec{x}_H(t) \to x_{tr}(t)$ (transient because of damping)

will be close to undamped soln, as computed in example above. Thus we can study practical resonance by computing how large the entries of $\vec{c}$ are, depending on ω.

MATH 2250-1
Mass-Spring systems
December 1, 2008

This maple document covers section 7.4 and might help for the Earthquake exploration you're doing at the end of that section. Here is how Maple computations work out the running example in today's notes.

```
> with(linalg):
> with(plots):with(DEtools):
> M:=matrix([[2,0],[0,1]]);
  K:=matrix([[-150,50],[50,-50]]);
  A:=evalm(inverse(M)&*K);
```

$$M := \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$K := \begin{bmatrix} -150 & 50 \\ 50 & -50 \end{bmatrix}$$

$$A := \begin{bmatrix} -75 & 25 \\ 50 & -50 \end{bmatrix}$$

today's notes page 2

```
> eigenvectors(A);
```

$$[-100, 1, \{[-1, 1]\}], [-25, 1, \{[1, 2]\}]$$

Therefore, the natural frequencies of this system are the 10 and 5, and the two fundamental modes correspond to the masses moving in opposite directions (with equal amplitudes and angular frequency 10) and in parallel directions (with amplitude ratio of two and angular frequency 5), as discussed on page 2 of today's handwritten notes.

Now, let's consider the forced system with force vector equal to cos(wt)[0,50], i.e. the second mass is being forced periodically. In other words, the system **Mx''=Kx + F**, where F=cos(wt)[0,50] ; this is Example 3 on page 327, and worked out in today's notes.

```
> F0:=evalm(inverse(M)&*vector([0,50])):    #The F0 in the
  normalized equation (30), page 436
  Iden:=array(1..2,1..2,identity):
      #the 2 by 2 identity matrix
  Aleft:=omega->evalm(A + omega^2*Iden):
      #the matrix function multiplying
      #c on the left side of (32)
  c:=omega->evalm(-inverse(Aleft(omega))&*F0):
      #the solution vector c(omega) to (32),
      #obtained by multiplying both sides of equation
      #(32) on the left, by the inverse to Aleft
> c(omega); #see equation (35) page 437,
      #and our hand work on page 3 of today's notes
```

page 3

$$\left[\frac{1250}{2500 - 125\,\omega^2 + \omega^4}, -\frac{50\,(-75 + \omega^2)}{2500 - 125\,\omega^2 + \omega^4}\right]$$

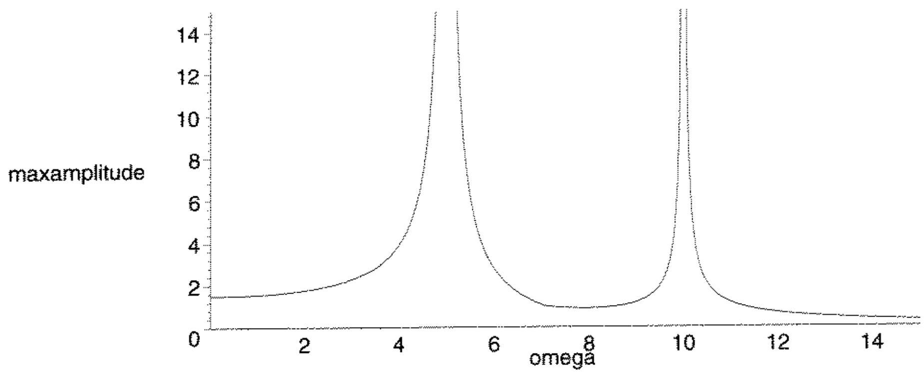
The vector c(w), as above, times the function of time, cos(wt), is a particular solution to the forced oscillation problem we are considering. If we assume that our actual problem has a small amount of damping, then we expect that this particular solution is very close to the steady periodic solution to the damped problem, as we just discuseed and also on pages 437-438 text. Study resonance phenomena for these slightly damped problems by plotting the maximum amplitude for the individual masses, in the steady state solutions to the undamped problems. Maple has the command ''norm'' to measure this maximum amplitude, ready for you to use in your Maple project.

```
> norm(c(omega));
```

$$\max\left(\frac{1250}{\left|2500 - 125\,\omega^2 + \omega^4\right|}, 50\left|\frac{-75 + \omega^2}{2500 - 125\,\omega^2 + \omega^4}\right|\right)$$

The following graph of the maximum amplitude norm for the undamped particular solution shows that in the slightly damped problem we expect practical resonance when omega is near 5 or 10 radians per second:

```
> plot(norm(c(omega)),omega=0..15,maxamplitude=0..15,
        numpoints=200,color='black');
```



This is qualitatively the picture on page 437, figure 7.4.10, although they plotted the (Euclidean) magnitude of c(omega) rather than the maximum of the individual amplitudes. Notice how we get Maple to label the axes as desired

We can get a plot of resonance as a function of period by recalling that 2*Pi/T=omega:

```
> plot(norm(c(2*Pi/period)),period=0.1..3,maxamplitude=0..15,
  numpoints=200,color='black');
```