Practice Exam #2 Solutions
Math 2250-3
November 10, 2004

1) Consider the homogeneous differential equation

$$deqtn := \left(\frac{d^2}{dt^2}\,\mathrm{x}(t)\right) + 8\left(\frac{d}{dt}\,\mathrm{x}(t)\right) + 20\,\mathrm{x}(t) = 0$$

1a) If this was modeling a mass-spring configuration like we studied in Chapter 5 of Edwards-Penney, and if the mass was 3 kg, what values of coefficient of friction and spring constant would lead to the differential equation above? (1 point for getting the units correct, 2 points for the correct numerical values).

(6 points)

*Since the general form of the unforced mass-spring systems is m*x'' + c*x' + k*x = 0 we see that we must have divided by the mass(=3) to get our deqtn above. So c must equal 24 and k must equal 60. Since the units of each term in the DE are units of force, the units of k in this case are newtons/meter (or kg/sec^2), and the units of c are newton sec/met (or kg/sec)*

1b) What kind of damping is exhibited by this mass-spring system?

(4 points)

*The characteristic equation is r^2 + 8*r +20 = 0, and the roots are*

```
> solve(r^2 + 8*r + 20=0,r);
```

$$-4 + 2\,I,\ -4 - 2\,I$$

So this is an underdamped system

1c) Find the general solution to this homogeneous differential equation

(5 points)

From the characteristic equation and Euler's forumula we deduce
xh(t) = exp(-4t)(c1*cos(2t) + c2*sin(2t))*

1d) Consider the same spring system, but now with a driving force F0(t)=9*cos(2t). Find the general solution to this inhomogeneous differential equation. Use the method of undetermined coefficients. Identify the steady periodic and transient pieces of the solution. Find the amplitude and phase of the steady periodic solution.

(20 points)

We try

```
> xp:=t->A*cos(2*t) + B*sin(2*t);
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$$xp := t \to A\cos(2\,t) + B\sin(2\,t)$$

```
> diff(xp(t),t,t)+8*diff(xp(t),t)+20*xp(t)=3*cos(2*t);
      #plug our guess into our deqtn.
      #Why is the rhs 3*cos... instead of 9*cos...?
      #(tricky huh?)
```

$$16\,A\cos(2\,t) + 16\,B\sin(2\,t) - 16\,A\sin(2\,t) + 16\,B\cos(2\,t) = 3\cos(2\,t)$$

```
> solve({16*A+16*B = 3, 16*B-16*A = 0},{A,B});
```

```
#equate coefficients
```

$$\{A=\frac{3}{32}, B=\frac{3}{32}\}$$

```
> xsp:=t->(3/32)*cos(2*t) + (3/32)*sin(2*t);
   #a particular solution, also our steady-periodic solution.
```

$$xsp := t \to \frac{3}{32}\cos(2\,t)+\frac{3}{32}\sin(2\,t)$$

```
> xtr:=t->exp(-4*t)*(c1*cos(2*t)+c2*sin(2*t));
    #transient solution
```

$$xtr := t \to \mathbf{e}^{(-4\,t)}\,(c1\cos(2\,t)+c2\sin(2\,t))$$

```
> x:=t->xsp(t)+xtr(t);
  x(t);
   #general solution
```

$$x := t \to \mathrm{xsp}(t)+\mathrm{xtr}(t)$$

$$\frac{3}{32}\cos(2\,t)+\frac{3}{32}\sin(2\,t)+\mathbf{e}^{(-4\,t)}\,(c1\cos(2\,t)+c2\sin(2\,t))$$

```
> deqtn2:=lhs(deqtn)=3*cos(2*t);
  dsolve(deqtn2,x(t));
    #check our work on Maple, since this is supposed
    #to be a solution key
```

$$deqtn2 := \left(\frac{d^2}{dt^2}\mathrm{x}(t)\right)+8\left(\frac{d}{dt}\mathrm{x}(t)\right)+20\,\mathrm{x}(t)=3\cos(2\,t)$$

$$\mathrm{x}(t)=\frac{3}{32}\sin(2\,t)+\frac{3}{32}\cos(2\,t)+\_C1\,\mathbf{e}^{(-4\,t)}\sin(2\,t)+\_C2\,\mathbf{e}^{(-4\,t)}\cos(2\,t)$$

```
> C:=sqrt(2*9)/32;
  a:=arctan(3/3);
  #amplitude and phase of steady state solution
```

$$C := \frac{3}{32}\sqrt{2}$$

$$a := \frac{1}{4}\pi$$

2) Here is a matrix:

```
> A:=matrix(3,5,[1,3,-4,-8,6,1,0,2,1,3,2,7,-10,-19,13]);
```

$$A := \begin{bmatrix} 1 & 3 & -4 & -8 & 6 \\ 1 & 0 & 2 & 1 & 3 \\ 2 & 7 & -10 & -19 & 13 \end{bmatrix}$$

Here is its reduced row echelon form:

```
> rref(A);
```

$$\begin{bmatrix} 1 & 0 & 2 & 1 & 3 \\ 0 & 1 & -2 & -3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

2a) Find a basis for the solution space (of homogeneous solutions) to Ax=0.

(10 points)

backsolving rref(a), augmented with a zero vector, we see that x5=t, x4=s, x3=r, x2=2r+3s-t, x1=-2r-s-3t, so

[x1,x2,x3,x4,x5]= r[-2,2,1,0,0] + s[-1,3,0,1,0] + t[-3,-1,0,0,1], so a basis for the nullspace is

```
> {[-2,2,1,0,0],[-1,3,0,1,0],[-3,-1,0,0,1]};
```

{[-2, 2, 1, 0, 0], [-1, 3, 0, 1, 0], [-3, -1, 0, 0, 1]}

2b) Explain what it means for a collection of vectors to be linearly dependent or linearly independent.

(5 points)

*The set {v1, v2, ..., vn} is dependent if a linear combination of them adds up to zero, other than the trivical linear combination for which all the coefficients ci are zero. The set is independent if the only way c1*v1 + c2*v2 + ... + cn*vn = 0 is when each ci=0.*

2c) Are the first three columns of A linearly independent or linearly dependent? If they are dependent, exhibit a dependency. If they are independent, explain why.

(10 points)

They are dependent. We can see this because column depencies are preserved when you do row operations (because they correspond to solutions of the homogeneous equation). In rref(A) we see that the third column is twice the first, minus twice the second. Thus it is also true that 2[1,1,2] - 2*[3,0,7] = [-4,2,-10].*

2d) Explain what it means for a collection of vectors to span a vector space.

(5 points)

*The set {v1, v2, ..., vn} spans the vector space V if and only if every element of V can be written as a linear combination of v1, v2, ..., vn, i.e. each v in V equals c1*v1 + c2*v2 + ... + cn*vn, for some choice of c1 thru cn.*

2e) Do the first three columns of A span all of R^3. Explain your answer.

(5 points)

They do not span all of R^3: Since the third column is dependent on the first two (see part 2c), the span of the three columns is the same as the span of the first two. [throwing dependent vectors away does not decrease the span of a set of vectors]. Since the first two columns of A are linearly independent, we deduce that they (and the original three columns) span a 2-dimensional subset of R^3, i.e. a plane through the origin.

2f) Find a basis for the rowspace of A.

(5 points)

Since elementary row operations do not change the span of the rows of a matrix, the row space is the span of the two non-zero rows in rref(A). Also, these rows are linearly independent, as is easy to check. Thus, a basis for the rowspace of A is {[1,0,2,1,3], [0,1,-2,-3,1]}.

3) Consider the differential equation

```
> deqtn:=diff(y(x),x,x,x) + 25*diff(y(x),x)= 10;
```

$$deqtn := \left(\frac{d^3}{dx^3}\, y(x)\right) + 25\left(\frac{d}{dx}\, y(x)\right) = 10$$

Find the solution to the initial value problem for this differential equation, with y(0)=4, D(y)(0)=0, D(D(y)(0))=10.

(25 points)

```
> dsolve({deqtn,y(0)=4,D(y)(0)=0,D(D(y))(0)=10},y(x));
    #Well, I can use Maple.  Of course you can't use it on the exam
```

$$y(x) = -\frac{2}{25}\sin(5\,x) - \frac{2}{5}\cos(5\,x) + \frac{2}{5}\,x + \frac{22}{5}$$

Here's how you would do this problem on the exam:

(1) find the homogeneous solutions:

The characteristic equation and its roots are

```
> r^3+25*r=0;
  solve(r^3+25*r=0,r);
```

$$r^3 + 25\,r = 0$$

$$0,\ 5\,I,\ -5\,I$$

So the solution to the homogeneous problem is

```
> yh:=x->c1 + c2*cos(5*x) + c3*sin(5*x);
```

$$yh := x \to c1 + c2\cos(5\,x) + c3\sin(5\,x)$$

(2) To find a particular solution we would guess a polynomial of degree zero, except that such things are solutions to the homogeneous equation, so using our recipe we guess

```
> yp:=x->A*x;
  diff(yp(x),x,x,x)+25*diff(yp(x),x)=10;
     #of course I could do this by hand
```

$$yp := x \to A\,x$$

$$25\,A = 10$$

So A=10/25=2/5

```
> y:=z->subs({A=2/5,x=z},yp(x))+yh(z);
     #general solution
  y(x);
```

$$y := z \to \mathrm{subs}\left(\left\{A = \frac{2}{5},\ x = z\right\},\ \mathrm{yp}(x)\right) + \mathrm{yh}(z)$$

$$\frac{2}{5}\,x + c1 + c2\cos(5\,x) + c3\sin(5\,x)$$

*(3) Solve the initial value problem. Computing y(0), D(y)(0) and D(D(y))(0) in terms of c1, c2, c3 and setting these equal to their IVP values leads to the system y(0) = c1 + c2 = 4, D(y)(0) = 2/5 + 5*c3 = 0,*

*D(D(y))(0) = -25*c2 = 10. This is easy to backsolve by hand, leading to c2=-10/25=-2/5, c3=-2/25, c1=4-c2=22/5. So our solution is*

```
> y1:=x->subs({c1=22/5, c2=-2/5, c3=-2/25,z=x},y(z));
  y1(x);
```

$$y1 := x \to \mathrm{subs}\left(\{z = x, c1 = \frac{22}{5}, c3 = \frac{-2}{25}, c2 = \frac{-2}{5}\}, \mathrm{y}(z)\right)$$

$$-\frac{2}{25}\sin(5\,x) - \frac{2}{5}\cos(5\,x) + \frac{2}{5}\,x + \frac{22}{5}$$

which is what Maple got at the start of the problem.