Math 2250-3
Practice Exam Solutions
October 2004

This exam is closed-book and closed-note. You may use a scientific calculator, but not one which is capable of graphing or of solving differential or linear algebra equations. **In order to receive full or partial credit on any problem, you must show all of your work** and **justify your conclusions.** There are 100 points possible. The point values for each problem are indicated in the right-hand margin. **Good Luck!**

1) Consider the following matrix equation, of the form Ax=b:

$$\begin{bmatrix} 2 & -1 & 2 \\ 1 & 1 & 1 \\ 1 & -5 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 3 \end{bmatrix}$$

1a) Compute the determinant of the coefficient matrix A above. What this tells you about the matrix equation and its possible solutions. In particular, can you rule out any of the three general possibilities: one solution, no solutions, or infinitiely many solutions, based on your determinant computation? Explain.

(8 points)

Using the "cross-hatch" method for 3 by 3 dets, for example, yields det(A)=2-1-10-2+10+1=0. Thus the matrix is not invertible. Thus rref(A) has a row of zeroes and a column without any leading 1, so we know that either our system is inconsistent (no solutions), or there are infinitely many (because there will be a free parameter in the geneal solution if the system is consistent). This means we have ruled out the possibility of a unique solution.

1b) Compute the reduced row echelon form of the 3 by 4 augmented matrix associated to the matrix equation above, and use it to solve the system.

(12 points)

Here's the augmented matrix:

$$Aaugb := \begin{bmatrix} 2 & -1 & 2 & 6 \\ 1 & 1 & 1 & 3 \\ 1 & -5 & 1 & 3 \end{bmatrix}$$

Here's the reduced row echelon form:

$$RREF := \begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

backsolving, we see z=t, y=0,x=3-t. In vector form this is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

2) Use the adjoint formula for the inverse matrix or Cramer's rule, to solve the following small but messy system of equations:

(10 points)

$$5\,x + 11\,y = 2$$
$$7\,x + 16\,y = 3$$

Since

$$A := \begin{bmatrix} 5 & 11 \\ 7 & 16 \end{bmatrix}$$

we have det(A)=80-77=3. Thus, using the adjoint formula, the inverse matrix is given by

$$Ainv := \frac{1}{3}\begin{bmatrix} 16 & -11 \\ -7 & 5 \end{bmatrix}$$

And

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 16 & -11 \\ -7 & 5 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \dfrac{-1}{3} \\ \dfrac{1}{3} \end{bmatrix}$$

3) Consider the differential equation

$$\frac{dP}{dt} = P^2 - 2\,P$$

which could be a model for a certain population problem.

3a) Find the equilibrium solutions.

(5 points)

*These are the constant solutions, so dP/dt = 0 = P^2-2P. Since*

$$P^2 - 2\,P = P\,[P - 2]$$

we deduce P=0, P=2 are the equilibrium solutions.

3b) Sketch the slope field for this differential equation. Onto the slope field sketch graphs of the solutions to the four initial value problems with P(0)=-1, P(0)=0, P(0)=1, P(0)=2, P(0)=3. (You don't need formulas for the solutions to make the sketches!)

When P<0 the slopes are positive, when 0<P<2 the slopes are negative, when P>2 the slopes are positive. For example, at P=-1 the slopes are 3, at P=1 they are -1 and at P=3 they are 3. This info enables one to draw the slope field picture. Then you add on the particular requested IVP graphs.
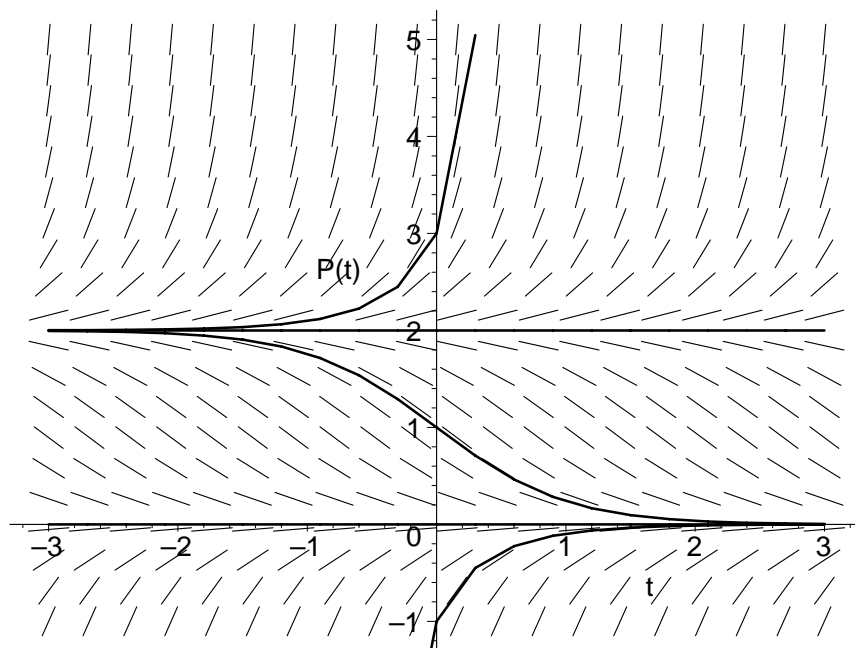
3c) Which of the equilibrium solutions are stable? Which are unstable?

(5 points)

Since solutions with IV near 0 converge to 0, P=0 is asymptotically stable. Since solutions with IV near 2 do not stay near 2, P=2 is unstable.

3d) Which of population models we studied would lead to differential equation of this type? Be as precise as you can, so that you account for the signs of both terms on the right of the differential equation.

(5 points)

This is a doomsday-extinction model, which could arise if the birth rate was proportional to P^2 and the death rate was proportional to P, for example.

3e) Find a explicit solution to the initial value problem for this differential equation, with P(0)=3. Explain what happens to your solution as time increases.

(15 points)

Since this is doomsday extinction and the initial population is greater than P=2, I expect doomsday! The DE is separable:

$$\frac{dP}{P(P-2)} = dt$$

We do partial fractions:

$$\frac{1}{P(P-2)} = \frac{A}{P} + \frac{B}{P-2}$$

$$1 = A(P-2) + BP$$

letting P=0 and then P=2 we see

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$$1 = -2A$$

$$1 = 2B$$

$$A = \frac{-1}{2}$$

$$B = \frac{1}{2}$$

Thus after multiplying both sides by 2, our separated DE is

$$\left(\frac{1}{P-2} - \frac{1}{P}\right)dP = 2\, dt$$

which integrates to

$$\ln\left(\left|\frac{P-2}{P}\right|\right) = 2t + C$$

exponentiate:

$$\left|\frac{P-2}{P}\right| = \mathbf{e}^C \mathbf{e}^{(2t)}$$

$$\frac{P-2}{P} = B\,\mathbf{e}^{(2t)}$$

at t=0, P=3, so we get

$$\frac{1}{3} = B$$

$$\frac{P-2}{P} = \frac{1}{3}\mathbf{e}^{(2t)}$$

$$P - 2 = \frac{1}{3}P\,\mathbf{e}^{(2t)}$$

$$P\left(1 - \frac{1}{3}\mathbf{e}^{(2t)}\right) = 2$$

$$P = 2\,\frac{1}{1 - \frac{1}{3}\mathbf{e}^{(2t)}}$$

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Note that at t=0 this simplifies to 2/(2/3)=3, as it should. As t increases the denominator shrinks to zero, reaching zero at

$$e^{(2t)} = 3$$

$$t = \frac{1}{2}\ln(3)$$

As t approaches this doomsday value the population approaches infinity.

4) Consider the following configuration of two brine tanks: The first tank holds 200 gallons of water, and the second tank holds 100 gallons. Solutions in each tank are well mixed so that each tanks' concentration is uniform. Pure water flows into the first tank, at a rate of 10 gallons per minute; water flows from tank 1 to tank 2 at the same rate, and water is removed from the second tank, also at the rate of 10 gallons per minute.

4a) Assuming that the first tank initially contained a brine solution in which there was 1/2 pounds of salt per gallon, explain why the amount of salt in this tank at time t minutes later is given by

$$x(t) = 100\,e^{(-0.05\,t)}$$

(10 points)

The governing equation is:

$$\frac{dx}{dt} = r_i\,c_i - r_o\,c_o$$

Since the incoming water is pure, the outgoing rate is 10 gallons/minute and concentration is x/200 pounds/gallon, this DE becomes

$$\frac{dx}{dt} = -\frac{1}{20}x$$

which we recognize as the decay equation from Calculus. Its solution is

$$x(t) = x_o\,e^{(-1/20\,t)}$$

Since the initial concentration is 1/2 pound per gallon, the initial amount is 100 pounds (in 200 gallons), so

$$x(t) = 100\,e^{(-0.05\,t)}$$

4b) Let y(t) be the amount of salt in the second tank. Use the formula for x(t) from 4a, and your modeling abilities to explain why y(t) satisfies the differential equation

$$\frac{dy}{dt} + 0.1\,y(t) = 5\,e^{(-0.05\,t)}$$

Again, the governing principle is

$$\frac{dy}{dt} = r_i\,c_i - r_o\,c_o$$

ri is 10, ci is x/200, so the first term is x/20. ro is 10 and co is y/100, so the second term is -y/10:

$$\frac{dy}{dt} = \frac{1}{20}x(t) - \frac{1}{10}y(t)$$

$$\frac{dy}{dt} = 5\,\mathbf{e}^{(-0.05\,t)} - 0.1\,y(t)$$

$$\frac{dy}{dt} + 0.1\,y(t) = 5\,\mathbf{e}^{(-0.05\,t)}$$

which is the claimed DE.

(8 points)

4c) Find the solution y(t) to the differential equation in 4b,, assuming the initial salt concentration in tank 2 is also 1/2 pound per gallon.

(12 points)

*This is a linear DEqtn; the integrating factor is exp(.1*t):*

$$\mathbf{e}^{(0.1\,t)}\left(\frac{dy}{dt} + 0.1\,y\right) = 5\,\mathbf{e}^{(0.1\,t)}\,\mathbf{e}^{(-0.05\,t)}$$

$$\left[\frac{d}{dt}\right]\mathbf{e}^{(0.1\,t)}\,y = 5\,\mathbf{e}^{(0.05\,t)}$$

$$\mathbf{e}^{(0.1\,t)}\,y = 100\,\mathbf{e}^{(0.05\,t)} + C$$

$$y = 100\,\mathbf{e}^{(-0.05\,t)} + C\,\mathbf{e}^{(-0.1\,t)}$$

Since y(0)=50 (pounds in 100 gallons),

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$$50 = 100 + C$$
$$C = \text{-}50$$
$$y = 100\,\mathbf{e}^{(-0.05\,t)} - 50\,\mathbf{e}^{(-0.1\,t)}$$