Math 2250-3

1st HW : due 9/1

1.1 3, 4, 7, 15, 16, 19, 20, 27, 30, 34, 36, 46

1.2 5, 7, 10, 14, 20, 25, 26, 36, 44

1.3 3, 6, 8, 11, 12, 14, 21, 25, 32, 33

1.1 #7) $y'' - 2y' + 2y = 0$

$y_1 = e^x \cos x$

$y_1' = e^x \cos x - e^x \sin x$

(product rule)

$y_1' = e^x(\cos x - \sin x)$

$y_1'' = e^x(\cos x - \sin x - \sin x - \cos x)$

$= e^x(-2\sin x)$

So $y_1'' - 2y_1' + 2y_1 = e^x\left[-\cancel{2\sin x} - 2(\cancel{\cos x} - \cancel{\sin x}) + 2\cancel{\cos x}\right]$

$= e^x(0) = 0$ ✓

$y_2 = e^x \sin x$

$y_2' = e^x(\sin x + \cos x)$

$y_2'' = e^x(\cancel{\sin x} + \cos x + \cos x - \cancel{\sin x})$

$= e^x(2\cos x)$

So $y_2'' - 2y_2' + 2y_2$

$= e^x\left[2\cancel{\cos x} - 2(\cancel{\sin x} + \cancel{\cos x}) + 2\cancel{\sin x}\right]$

$= e^x(0) = 0$ ✓

16) $3y'' + 3y' - 4y = 0$

try $y = e^{rx}$

$y' = re^{rx}$

$y'' = r^2 e^{rx}$

so $3y'' + 3y' - 4y = e^{rx}(3r^2 + 3r - 4)$

$= e^{rx}$ (doesn't factor nicely!)

so, roots of $3r^2 + 3r - 4 = 0$ by quadratic formula

$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2}$;

$r = \frac{-3 \pm \sqrt{9 - 4(3)(-4)}}{6} = \boxed{-\frac{1}{2} \pm \frac{\sqrt{57}}{6}}$

20) $y' = x - y$

$y = Ce^{-x} + x - 1$

is sol'n?

$y' = -Ce^{-x} + 1$

$x - y = \cancel{x} - [Ce^{-x} + \cancel{x} - 1] = -Ce^{-x} + 1$

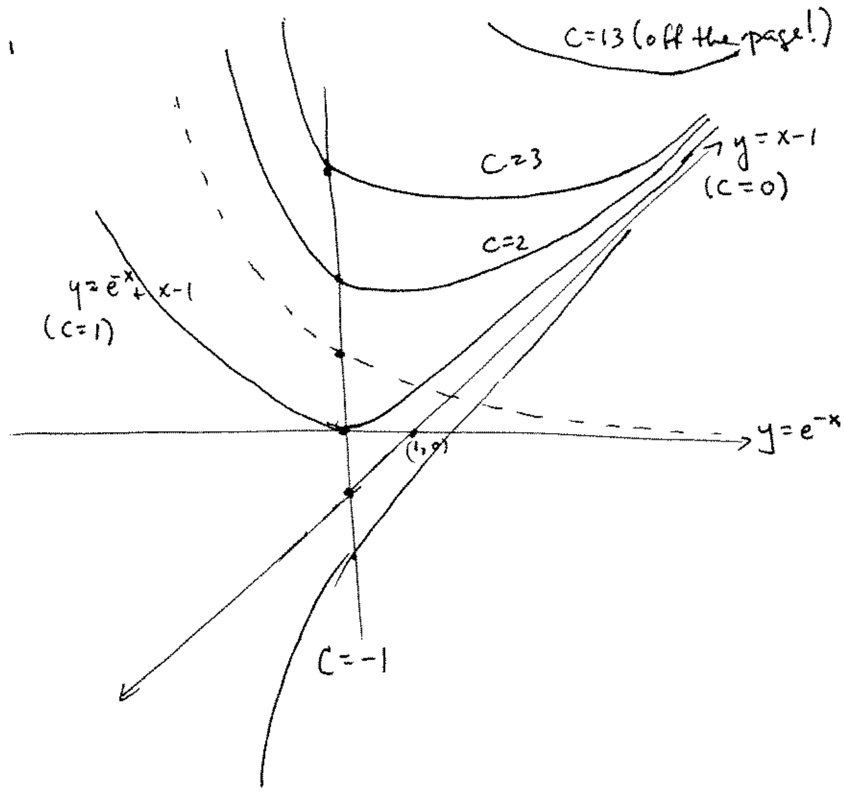
equal! So y is a solution

$$\begin{cases} y' = x - y \\ y(0) = 10 \end{cases}$$

$y(0) = 10 = C + 0 - 1$

$\Rightarrow C = 11$

$\boxed{y(x) = 11e^{-x} + x - 1}$



§1.1 #30) $y = g(x)$

parabola $y = x^2 + k$ has slope $\frac{dy}{dx} = 2x$ at every point (x, y) on its graph

$\perp$ lines have negative reciprocal slopes

So $y = g(x)$ satisfies $\boxed{\frac{dy}{dx} = -\frac{1}{2x}}$

34) $\frac{dv}{dt} = \underbrace{k \cdot}\ (\underbrace{250 - v})$

$\uparrow$ acceleration, $\uparrow$ is proportional to, the difference between 250 and v

36) $\frac{dN}{dt} = \underbrace{k \cdot}\ \underbrace{(N)(P-N)}$

$\uparrow$ time rate of change of # $N(t)$ of infected people

$\uparrow$ is proportional to

the product of # infected with # not infected

(P = # of people in town population)

46) $\begin{cases} \frac{dv}{dt} = kv^2 \\ v(0) = 10 \text{ m/sec.} \end{cases}$

and

$\frac{dv}{dt} = -1$ m/sec when $v = 5$ m/sec.

so from DE

$-1 = k \cdot 25$

So $k = -\frac{1}{25}$

Sol'n by separating variables:

$\frac{dv}{dt} = -\frac{v^2}{25}$

$-\frac{dv}{v^2} = \frac{1}{25}\,dt$

$\int \quad \int$

$\boxed{\frac{1}{v} = \frac{1}{25}t + C}$

$v = 0 = 10$

$\frac{1}{10} = 0 + C$

$\frac{1}{v} = \frac{1}{25}t + \frac{1}{10}$

$\frac{25}{v} = t + 2.5$

$\frac{v}{25} = \frac{1}{t+2.5}$; $\boxed{v = \frac{25}{t+2.5}}$

46a) $v(t) = 1$

$1 = \frac{25}{t+2.5} \Rightarrow \boxed{t = 22.5 \text{ sec}}$

46b) $v(t) = \frac{1}{10} = \frac{25}{t+2.5}$

$t + 2.5 = 250$

$\boxed{t = 247.5 \text{ sec}}$

46c) Never stops;

$v(t) \searrow 0$ as $t \to \infty$

§1.2 #7 $\begin{cases} \frac{dy}{dx} = \frac{10}{x^2+1} \\ y(0) = 0 \end{cases}$

$y = \int \frac{10}{x^2+1}\,dx = 10 \arctan x + C$

$y(0) = 0 = 10\cdot 0 + C; \quad C = 0$

$\boxed{y = \tan^{-1} x}$

10) $\begin{cases} \frac{dy}{dx} = xe^{-x} \\ y(0) = 1 \end{cases}$

use integrable table in book cover, #46, or integrate by parts

$y = \int xe^{-x}\,dx + C \quad = \int u\,dv = uv - \int v\,du$

$u = x$
$dv = e^{-x}dx$
$v = -e^{-x}$

$= -xe^{-x} - \int -e^{-x}\,dx$

$y = -xe^{-x} - e^{-x} + C$

$y(0) = 1 = -1 + C; \quad C = 2$

$$\boxed{\begin{aligned} y &= -xe^{-x} - e^{-x} + 2 \\ &= 2 - e^{-x}(x+1) \end{aligned}}$$

14) $a(t) = 2t + 1$
$V_0 = -7$
$x_0 = 4$

Find $x(t)$:

$v(t) = \int a(t)\,dt$
$= t^2 + t + C$
$v(0) = -7 = C$
$v(t) = t^2 + t - 7$

$x(t) = \int v(t)\,dt = \frac{t^3}{3} + \frac{t^2}{2} - 7t + C$
$x(0) = 4 = C$

$\boxed{x(t) = \frac{t^3}{3} + \frac{t^2}{2} - 7t + 4}$

20) (from picture)

$v(t) = \begin{cases} t & 0 \le t \le 5 \\ 5 & t \ge 5 \end{cases}$

$x(0) = 0$, so $x'(t) = t \quad 0 \le t \le 5$

$\Rightarrow \boxed{x(t) = \frac{1}{2}t^2 \quad 0 \le t \le 5}$

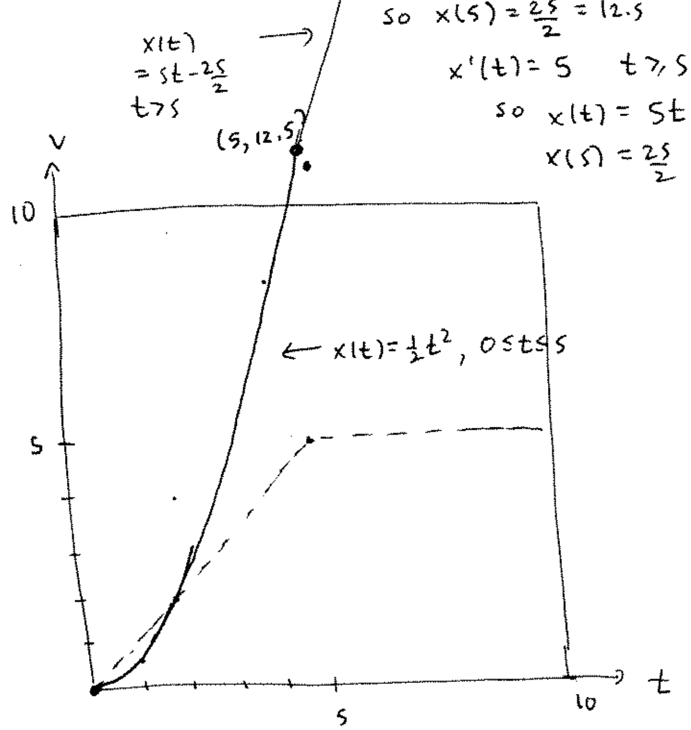
so $x(5) = \frac{25}{2} = 12.5$

$x'(t) = 5 \quad t \ge 5$

so $x(t) = 5t + C$

$x(5) = \frac{25}{2} = 25 + C; \quad C = -\frac{25}{2}$

$x(t) = \begin{cases} \frac{1}{2}t^2 & 0 \le t \le 5 \\ 5t - \frac{25}{2} & t \ge 5 \end{cases}$



don't actually need formula to sketch → since the velocity curve *is* the slope of the graph!

1.2 #25) $\begin{cases} \frac{dv}{dt} = -10 \quad m/sec^2 \\ v(0) = \frac{10^2}{3.6} \quad m/sec \end{cases}$ $\quad v(0) = 100 \ km/h$

$$= 100 \frac{km}{h} \cdot 10^3 \frac{m}{km} \cdot \frac{1 \ h}{60^2 \ sec} = \frac{10^5}{3.6 \times 10^3} \ m/sec$$

$$= \frac{10^2}{3.6} \ m/sec.$$

so $v(t) = -10t + C$

$v(0) = C$

$v(t) = -10t + \frac{10^2}{3.6}$ as long as $v(t) \geq 0$

$x(t) = -5t^2 + \frac{10^2}{3.6} t + \cancel{C}$

$v(t) = 0$ when $10t = \frac{10^2}{3.6}$; $t = \frac{10}{3.6}$ sec. ≈ 2.78 sec

at this instant car has traveled in pos. direction,

distance $x\left(\frac{10}{3.6}\right) = -5\left(\frac{10}{3.6}\right)^2 + \frac{10^2}{3.6}\frac{10}{3.6}$

$= \left(\frac{10}{3.6}\right)^2 [-5 + 10] = \frac{500}{(3.6)^2}$ $\boxed{\approx 38.6 \text{ meters}}$

#26) $x''(t) = -g = -9.8 \ m/sec^2$

(positive ht $x(t)$ "up").

so $x'(t) = -9.8t + v_0$

$x'(t) = -9.8t + 100$ m/sec

so $x(t) = -4.9t^2 + 100t + x_0$

$= -4.9t^2 + 100t + 20$

(a) Max ht when $v(t) = x'(t) = 0$

ie $9.8t = 100$

$\boxed{t = \frac{100}{9.8} \cong 10.20 \text{ sec}}$

max ht $\cong x(10.2) \cong \boxed{530 \ m}$

(b) $x(t) = 20$

$-4.9t^2 + 100t + \cancel{20} = \cancel{20}$

$t(-4.9t + 100) = 0$ $\boxed{t = 0, \ t = \frac{100}{4.9} \cong 20.41 \text{ sec}}$

(c) $x(t) = 0$

$-4.9t^2 + 100t + 20 = 0$

$t = -.198$, $\boxed{20.61 \text{ sec}}$ (Maple, or quadratic formula)

36) In general

$x(t) = -\frac{1}{2}at^2 + v_0 t + x_0$

$= -\frac{1}{2}at^2 + v_0 t$ if a is the accel of gravity (wherever), and $x_0 = 0$

& $v(t) = -at + v_0$

max ht when $v(t) = 0 = -at + v_0 \Rightarrow t = \frac{v_0}{a}$

& $x\left(\frac{v_0}{a}\right) = -\frac{1}{2}a\left(\frac{v_0}{a}\right)^2 + v_0\left(\frac{v_0}{a}\right) = \boxed{\frac{1}{2}\frac{v_0^2}{a}}$ is max height

assume same v_0 on earth & moon.

On earth $2.25 \cdot ft = \frac{1}{2}\frac{v_0^2}{a} \Rightarrow v_0^2 = 2a(2.25)$

$v_0^2 = 2 \cdot 32 \cdot \frac{9}{4} = 16 \cdot 9$; $\boxed{v_0 = 12 \text{ m/sec}}$

36 cont'd.

So on moon, with $a \approx 5.3$ ft/sec^2

$$\text{max ht} = \frac{1}{2}\frac{v_0^2}{a} = \frac{1}{2}\frac{12^2}{5.3} \cong \boxed{13.58 \text{ ft}}$$

44) const acceleration (deceleration),

so

$$x''(t) = -a$$
$$x'(t) = -at + v_0$$
$$x(t) = -\tfrac{1}{2}at^2 + v_0 t + 0 \qquad (\text{if } x_0 = 0)$$

police deduce a from:

with $v_0 = 25$

$$\underbrace{x'(t) = 0 \text{ when } x(t) = 45}$$

$$-at + v_0 = 0$$

$$t = \frac{v_0}{a} \quad ; \quad x\left(\frac{v_0}{a}\right) = 45$$

$$-\tfrac{1}{2}a\left(\frac{v_0}{a}\right)^2 + v_0\left(\frac{v_0}{a}\right) = 45 \qquad \text{(looks like 36!)}$$

$$\left.\begin{array}{l} \tfrac{1}{2}\frac{v_0^2}{a} = 45 \\ v_0 = 25 \end{array}\right\} \Rightarrow a = \frac{v_0^2}{90} = \frac{25^2}{90} = \frac{125}{18} \approx 6.94$$

What is v_0 if if

$x'(t) = 0$ when $x(t) = 210$?

$$210 = -\tfrac{1}{2}at^2 + v_0 t \quad \text{when } t = \frac{v_0}{a}$$

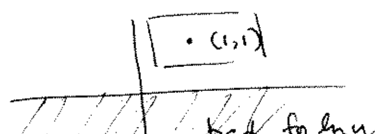
$$210 = -\tfrac{1}{2}a\left(\frac{v_0}{a}\right)^2 + v_0\left(\frac{v_0}{a}\right) = \tfrac{1}{2}v_0^2/a$$

$$\Rightarrow v_0^2 = 210 \cdot a \cdot 2$$

$$v_0 = \sqrt{420a} = \sqrt{420 \cdot \frac{125}{18}} \cong \boxed{54 \text{ miles/hour}} \quad \text{(actually exact)}$$

§1.3 6, 8 → see back of book!

12) $\begin{cases} \frac{dy}{dx} = x \ln y \\ y(1) = 1 \end{cases}$



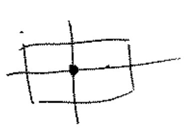
if rectangle is in upper half plane, Thm 1 applies

since $\left.\begin{array}{l} f(x,y) = x\ln y \\ D_y f = x/y \end{array}\right\}$ both continuous in rect

$\Big\}$ existence and unique on some interval containing $x=1$.

(actually $y \equiv 1$ is unique soln for all x)

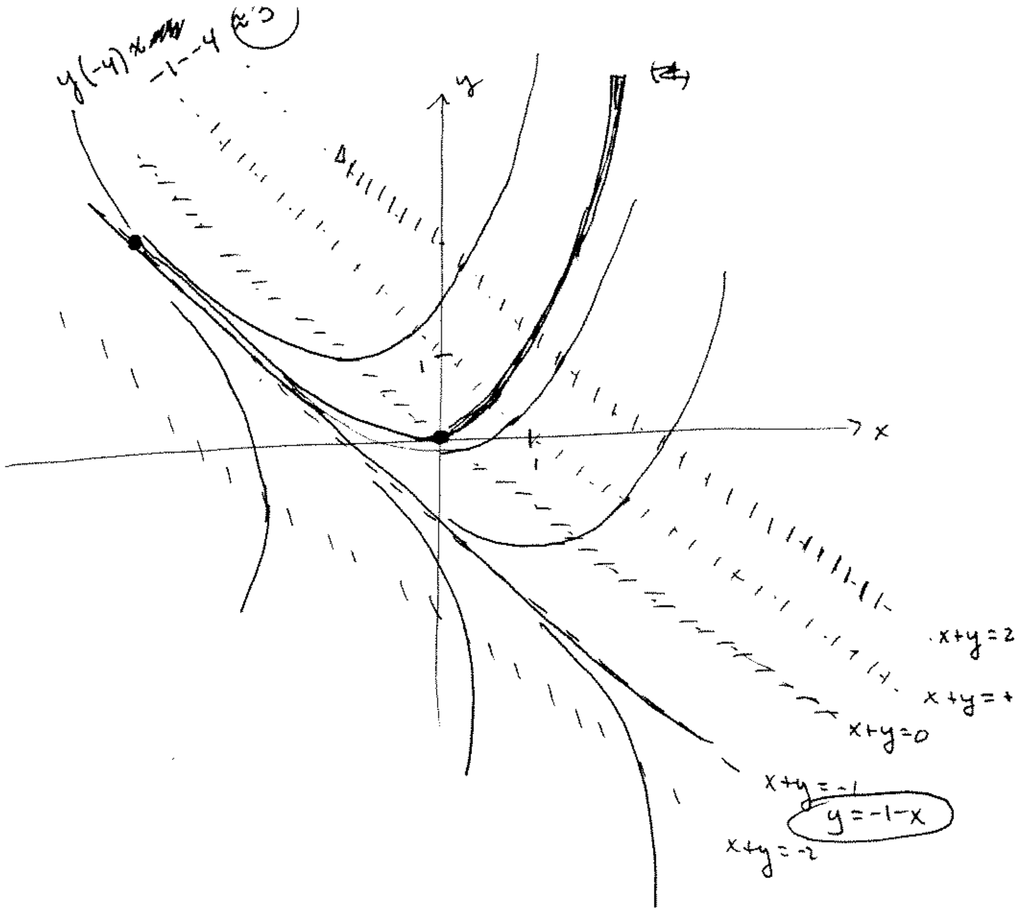
14) $\begin{cases} \frac{dy}{dx} = y^{1/3} \\ y(0) = 0 \end{cases}$



$\left.\begin{array}{l} f(x,y) = y^{1/3} \text{ cont on } \mathbb{R}^2 \\ D_y f = \frac{1}{3}y^{-2/3} \to \text{bad along x-axis } (y=0) \\ \quad \text{(not even defined)} \end{array}\right\}$ Thm 1 does NOT apply.

(In fact $y \equiv 0$ is a soltn but there are others)

(21) $\begin{cases} y' = x+y \\ y(0)=0 \end{cases}$



curve	slope
$x+y=0$	0
$x+y=1$ ($y=-x+1$)	1
$x+y=2$	2
$x+y=-1$	-1

(25) $\begin{cases} \dfrac{dv}{dt} = 32 - 1.6v \\ v(0)=0 \end{cases}$

limiting velocity is from $\frac{dv}{dt} = 0 = 32 - 1.6v$;

$v = \frac{32}{1.6} = 20$ ft/sec

$= 20 \text{ ft/sec} \cdot \frac{1 \text{ mi}}{5280 \text{ ft}} \cdot \frac{3600 \text{ sec}}{1 \text{ h}}$

$\cong 13.6$ mi/hour (not bad) (haystack nice but not necessary)

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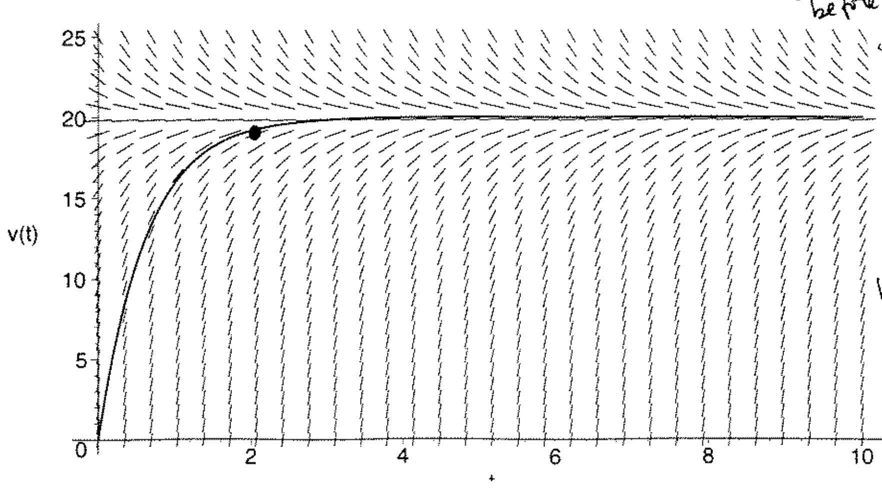
> restart:with(plots):with(DEtools):
> deqtn:=diff(v(t),t)=32-1.6*v(t):
  dsolve({deqtn,v(0)=0},v(t));  #Maple solution
  DEplot(deqtn,v(t),t=0..10,{[v(0)=0]},v=0..25,arrows=line,
  color=black,linecolor=black,dirgrid=[30,30],stepsize=.1);
```

Warning, the name changecoords has been redefined

$$v(t) = 20 - 20\,e^{\left(-\frac{8t}{5}\right)}$$



you approach limiting velocity very fast, long before set to ground. (95%)(20) ≈ 19 ft/s at about $t=2$ sec, before you've even fallen 40 ft!