

Name.....
I.D. number.....

Math 2250-3
PRACTICE FINAL EXAM

December 2004

(This was once an actual final exam)

This exam is closed-book and closed-note. You may use a scientific calculator, but not one which is capable of graphing or of solving differential or linear algebra equations. **In order to receive full or partial credit on any problem, you must show all of your work and justify your conclusions.** This exam counts for 30% of your course grade. It has been written so that there are 200 points possible, however, and the point values for each problem are indicated in the right-hand margin. **Good Luck!**

1) Consider the initial value problem

$$\frac{dy}{dx} + 3y = 2x$$
$$y(0) = 1$$

1a) Solve this problem using Chapter 1 techniques. (By the way, there will be an integral table and a Laplace transform table for the actual final exam.)

(10 points)

1b) Since the differential equation we are considering in this problem is a constant-coefficient linear one, the methods of Chapter 5 also apply. Explain how you would go about finding the general solution to the differential equation if you were using those techniques. In particular, how would you find the general solution to the homogeneous equation (and what is it)? What type of particular solution would you try to find?

(10 points)

2) Consider the differential equation

$$\frac{dP}{dt} = -P^2 + 2P$$

which models a certain logistic population problem.

2a) Find the equilibrium solutions.

(4 points)

2b) Sketch the phase portrait and the slope field for this differential equation. Onto the slope field sketch graphs of the solutions to the four initial value problems with $P(0)=0$, $P(0)=1$, $P(0)=2$, $P(0)=3$ (You don't need formulas for the solutions to make the sketches!)

(7 points)

2c) Which of the equilibrium solutions are stable? Which are unstable?

(4 points)

2d) Find an explicit solution to the initial value problem for this differential equation, with $P(0)=1$. Verify that your limiting population agrees with what your sketch predicted in part 1b).

(15 points)

3) Consider the homogeneous differential equation

$$\frac{d^2 x}{dt^2} + 8 \frac{dx}{dt} + 20 x = 0$$

3a) If this was modeling a mass-spring configuration like we studied in Chapter 5 of Edwards-Penney, and if the mass was 3 kg, what values of coefficient of friction and spring constant would lead to the differential equation above? (1 point for getting the units correct, 2 points for the correct numerical values).

(6 points)

3b) What kind of damping is exhibited by this mass-spring system?

(4 points)

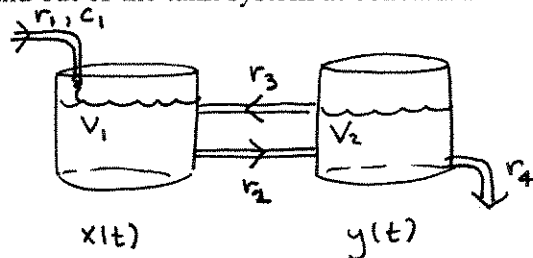
3c) Solve the initial value problem for the differential equation above, where $x(0)=5$ and $dx/dt(0)=4$. Use the methods of Chapter 5.

(20 points)

3d) Re-solve the initial value problem of 3c), this time using the Laplace Transform techniques of Chapter 10. Of course, your answers to 3c) and 3d) should agree if you do both parts correctly.

(20 points)

4) Consider the following two-tank configuration. In tank one there is uniformly mixed volume of V_1 gallons, and pounds of solute $x(t)$. In tank two there is mixed volume of V_2 gallons and pounds of solute $y(t)$. Water is pumped into tank one at a constant rate of r_1 gallons/minute from an outside source, and this water has a constant solute concentration of c_1 pounds/gallon. Water is pumped from tank one to tank two at constant rate of r_2 gallons/minute, from tank two to tank one at constant rate r_3 gallons/minute, and out of the tank system at constant rate r_4 gallons/minute.



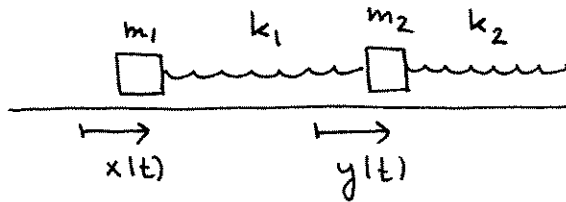
4a) What conditions on the rates r_1, r_2, r_3, r_4 are necessary to guarantee that the volumes V_1 and V_2 remain constant in time?

(5 points)

4b) Write the system of first order differential equations which governs the process described above. Do not try to solve these DE's.

(10 points)

7) Consider the following configuration of springs, with positive displacements from equilibrium measured to the right, as indicated.



7a) Derive the system of second order differential equations which models this system. Assume that there are no external forces.

(5 points)

7b) Assume that in appropriate units $m_1=2$, $m_2=2$, $k_1=4$, $k_2=6$. Show that in this case your system above reduces to

$$\begin{bmatrix} \frac{d^2 x}{dt^2} \\ \frac{d^2 y}{dt^2} \end{bmatrix} = \begin{bmatrix} -2x + 2y \\ 2x - 5y \end{bmatrix}$$

(5 points)

7c) Find the general solution to the unforced system (7b).

(15 points)

7d) Assuming ω is not a natural frequency for the problem above, find a particular solution to the forced system

$$\begin{bmatrix} \frac{d^2 x}{dt^2} \\ \frac{d^2 y}{dt^2} \end{bmatrix} = \begin{bmatrix} -2x + 2y + \cos(\omega t) \\ 2x - 5y - \cos(\omega t) \end{bmatrix}$$

(10 points)

5) Let A be the matrix:

$$A := \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

not invertible
invertible

- 5a) Compute the determinant of A and use it to determine whether A is singular or nonsingular. (5 points)
- 5b) Does your answer to (5a) let you deduce the reduced row echelon form of A? Explain. (5 points)
- 5c) Exhibit a basis for the column space of A. Explain. (5 points)
- 5d) Find the inverse matrix to A. (10 points)

6) Let B be the matrix

$$B := \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -1 & 1 & 4 \end{bmatrix}$$

- 6a) Find a basis for \mathbb{R}^3 made out of eigenvectors of B (20 points)
- 6b) Use your answer from part 6a) to write down the general solution to the system $dx/dt = Bx$, of three first order differential equations. (5 points)