Due date: See the internet due dates.

References: Edwards-Penney, pages 55–57. Code in `maple` appears in `2250mapleL1b-F2004.txt` at URL
`http://www.math.utah.edu/~gustafso/`. This document: `2250mapleL1b-F2004.pdf`.

The Problem. The project is to determine an approximation for the indoor temperature $u(t)$ in an unheated building. The model uses Newton cooling, insulation data and a formula for the ambient outside temperature $A(t)$.

Assumptions and Notation. Let the daily temperature in Salt Lake City vary from a low of $m = 20$ degrees to a high of $M = 46$ degrees with the low and high at 3am and 3pm, respectively. The building is assumed insulated, the average insulation factor being represented by a Newton cooling constant k, usually $0.2 \le k \le 0.5$. The ambient outside temperature is called $A(t)$ and the inside temperature is called $u(t)$, where t is in hours measured from midnight $t = 0$. The number u_0 is the temperature at midnight. The model assumes that no heat sources or heat sinks are present inside the building.

Newton Cooling Model. Newton's law of cooling is:

The rate of change of the indoor temperature is proportional to the difference between the ambient and indoor temperatures.

This implies that du/dt is proportional to $A - u$. Let $A(t)$ be the ambient outside temperature and let k denote a constant. A model for the indoor temperature $u(t)$ is given by $du/dt = k(A - u)$, $u(0) = u_0$, which can be rearranged as

(1) $$u'(t) + ku(t) = kA(t), \quad u(0) = u_0.$$

The number k is called the **insulation constant**.

Ambient Temperature Model. Let $M = 46$, $m = 20$, $\omega_0 = \pi/12$. The ambient temperature formula

$$A(t) = \frac{1}{2}(M + m) - \frac{1}{2}(M - m)\cos\omega_0(t - 3)$$

satisfies $M = \max A(t) = A(15)$, $m = \min A(t) = A(3)$ and $A(t)$ is t-periodic of period 24 hours.

For use in `maple`, a function of two variables

$$AA(t,\omega) = \frac{1}{2}(M + m) - \frac{1}{2}(M - m)\cos\omega(t - 3)$$

is used to maintain the variable name ω in displays.

Indoor Temperature $u(t)$. The integrating factor method for linear equations applies to find the general solution by these steps:

$u' + ku = kA(t)$	Copy the differential equation. The integrating factor is e^{kt}.
$\dfrac{(e^{kt}u)'}{e^{kt}} = kA(t)$	Replace the left side.
$(e^{kt}u)' = kA(t)e^{kt}$	Clear fractions.
$e^{kt}u = u_0 + \int_0^t kA(x)e^{kx}\,dx$	Integrate both sides with respect to t. Apply the Fundamental Theorem of Calculus. Use $u(0) = u_0$.
$u = u_0 e^{-kt} + e^{-kt}\int_0^t kA(x)e^{kx}\,dx$	Divide to isolate u.

Let $u_h(t) = u_0 e^{-kt}$, a solution of the homogeneous differential equation $u' + ku = 0$. Let $u_p(t) = ke^{-kt}\int_0^t e^{kx}A(x)dx$, a particular solution of the nonhomogeneous differential equation $u' + ku = kA(t)$. Then the indoor temperature $u = u_h + u_p$ depends on the time t, the initial temperature u_0, the insulation constant k and the frequency ω (which is fixed at $\pi/12$). Write $u = u(t, u_0, k, \omega)$ to emphasize the dependence. In `maple`, advantages exist for adding the variable name ω, which is later set to value $\omega_0 = \pi/12$. Write u as $\boxed{\texttt{U(t,u0,k,omega)}}$ for use in `maple`.

Steady-state solution. The steady-state solution u_{SS} is obtained from the general formula $u = u_h + u_p$ by dropping all terms containing a negative exponential. It depends on t, k and ω but it is independent of u_0.

Problem 1.3. (Solution formulas for u_p and u)

Derive by hand, using integral tables, an explicit symbolic formula for $u_p(t)$. Display a final formula for $u = u_h + u_p$ which depends only on t, u_0, k and ω. Check your hand answer for u in `maple`. The only `maple` assist in this problem is the answer check.

Problem 1.4. (Steady-state Periodic Solution)

Derive by hand a formula for the steady-state periodic solution u_{SS} of $u' + ku = kA(t)$. The only `maple` assist in this problem is an answer check.

Problem 1.5. (Indoor-Outdoor Variation)

Compare in a `maple` graphic the indoor ($u(t)$) and outdoor ($A(t)$) temperature oscillations over a 48-hour period assuming $k = 0.35$, $u_0 = 72$, $\omega = \pi/12$. Compute the indoor and outdoor temperature variations from this 48-hour plot. Find the phase delay from this same plot.

Problem 1.6. (Freezing Pipes)

Assume $\omega = \pi/12$ and the insulation constant k ranges from 0.2 to 0.4. Suppose the inside temperature is 72 degrees at midnight when the furnace is turned off. Report approximate ranges of hours during the first 72 hours, for which the indoor temperature is at or below 30 degrees. Justify your logic used to find the ranges, in a short paragraph. Illustrate with a computer graphic.

Problem notes.

Notes on 1.3: The integration problem to be solved by hand using the book's integral tables is

$$u_p(t) = ke^{-kt} \int_0^t e^{kx}(33 - 13\cos\omega(x-3))dx.$$

A change of variable $u = x - 3$ allows use of integral table entry $\int e^{au}\cos bu\, du$. The symbol ω will be set to $\pi/12$, but for simplicity, use symbol ω throughout.

The formulae for u_h and u_p are used again in 1.4 in order to derive the steady-state solution. Your answer *must contain* symbols $\boxed{\texttt{t, u0, k, omega}}$.

The answer check in `maple` is organized as follows. The complications of setting $\omega = \pi/12$ are avoided here by leaving ω as a symbol, since it does not affect the answer check.

```
# Test LHS=RHS for u'+ku=kA.
t:='t': u0:='u0': omega:='omega': k:='k':
myANS:=your hand-derived formula for u=uh+up:
LHS:=diff(myANS,t)+k*myANS:
RHS:=k*(33-13*cos(omega*(t-3))):
simplify(expand(LHS-RHS));
```

A successful test of $LHS = RHS$ produces answer *zero*, or an expression that reduces to zero.

The algebra system `maple` can verify table use, for example, $\boxed{\texttt{int(exp(k*x)*cos(w*(x-3)),x=0..t)}}$ will check some of the hand integration details.

Notes on 1.4: The steady-state solution is derived from the hand-generated symbolic solution $u = u_h + u_p$ in 1.3 by dropping all terms that contain e^{-kt}. The answer, where $\omega_0 = \pi/12$:

$$u_{SS} = 33 - \frac{13k}{k^2 + \omega_0^2}(k\cos\omega_0(t-3) + \omega_0\sin\omega_0(t-3)).$$

To *check your answer*, use `maple` as in 1.3.

To get `maple` to report the above formula, it is essential to evaluate everything with ω as a *symbol*, to wit, use in `maple` the statement $\boxed{\texttt{omega:='omega':}}$. **Beware** of writing $\boxed{\texttt{omega:=Pi/12}}$, which defines ω to be a constant, unless you undo the effect immediately afterward. Maple's $\boxed{\texttt{unassign}}$ can be used to erase assignments.

Notes on 1.5: The outside temperature $A(t)$ (`AA(t,omega)` in `maple`) and the solution $u = U(t, u_0, k, \omega)$ obtained in 1.3 can be programmed in `maple` as follows:

```
with(plots): unassign('t','u0','k','omega'):
AA:=(t,omega)->33-13*cos(omega*(t-3));
U:=(t,u0,k,omega)-> your answer of u=uh+up from 1.3:
```

The two curves are placed onto the **plot** by this `maple` command:

```
plot({U(t,72,0.35,Pi/12),AA(t,Pi/12)},t=0..48);
```

Missing curves? Probably, the missing curves are defined to contain an unevaluated variable name, like $\boxed{\texttt{pi}}$ instead of $\boxed{\texttt{Pi}}$. While `Pi` is the constant 3.14159, `pi` is a variable name: *case* is significant in `maple`. Curves vanish on the printer? Add $\boxed{\texttt{color=black}}$ to the plot command.

Click a mouse button on the high and low spots in the graphic. Somewhere on the `maple` worksheet the coordinates of the click are displayed. This method gives decimal approximations to the max and min values.

The indoor temperature variation is just the maximum minus the minimum, as computed from the plot of $u(t)$. Beware: u has a maximum at $t = 0$. Similarly, the outdoor temperature variation is computed from the graph of $A(t)$.

The phase shift is computed as $|T_2 - T_1|$, where $A(T_1) = \max A(t)$ and $u(T_2) = \max u(t)$. Look at the graphic to find sane answers for T_1 and T_2. See the textbook for a more complete discussion of the ideas. Beware: T_1 and T_2 are values for time in the first 72 hours (abscissa values, not ordinate values). The smallest phase shift obtainable from the graphic is to be reported, because the transient solution $u_h(t)$ may affect the graph of $u(t) = u_h(t) + u_{ss}(t)$ for a few hours.

Notes on 1.6: A computer algebra assist for this problem can be found in `maple`'s function `implicitplot`. This function can plot the equation $u(t, 72, k, \pi/12) = 30$ over the domain $0 \le t \le 72$, $0.2 \le k \le 0.4$. From this plot, and the 3D-plot $z = u(x, 72, y, \pi/12)$, the question is easily answered.

```
with(plots): unassign('t','u0','k','omega'):
U:=(t,u0,k,omega)->your answer of uh+up from 1.3:
implicitplot(U(t,72,k,Pi/12)=30,t=0..72,k=0.2..0.4);
plot3d({U(t,72,k,Pi/12),30},t=0..72,k=0.2..0.4);
```

Zoom in on the implicit plot by using a smaller time domain, suggested by the larger plot. The relation between the implicit plot and the 3D plot is seen by slicing the 3D plot at height $z = 30$ to obtain a *bread slice* parallel to the xy-plane. The implicit plot depicts the bread slice by projecting it onto the xy-plane.

Physically, inside temperature 30 degrees is reached a few hours *after* the outside temperature drops below 30 degrees. During 72 hours, there are three such inside temperature drops (see the 3D-plot, where z=temperature). For $k < 0.26$, inside temperature 30F is not reached in the first 23 hours. It is easiest to give two reports, one for $0.26 < k < 0.4$ and one for $0 < k < 0.26$. Report answers like $\boxed{\texttt{9-12, 37.5-39.2, 56-59 hours for } 0.26 < k < 0.4}$, plus an explanation of the logic applied to obtain this answer from the graphics.