

## Math 2250-3

### Notes for Separable Differential Equations - %1.4

Wednesday August 27, 2003

A differential equation

$$\left[ \frac{dy}{dx} = h(x, y) \right]$$

is called **separable** iff  $h(x,y)$  is a product of a function of  $x$  times a function of  $y$ ,

$$\left[ \frac{dy}{dx} = g(x) \phi(y) \right]$$

This is equivalent to the DE

$$\left[ \frac{dy}{dx} = \frac{g(x)}{f(y)} \right]$$

where  $f$  and  $\phi$  are reciprocal functions.

#### How to solve:

The algorithm is very simple, but magic: treat  $dy/dx$  as a quotient of differentials (!), and multiply through to rewrite the DE as

$$\left[ f(y) dy = g(x) dx \right]$$

[ Then antidifferentiate the left side with respect to  $y$  and the right side with respect to  $x$ .

$$\left[ \int f(y) dy = \int g(x) dx + C \right]$$

[ If  $F(y)$  and  $G(x)$  are antiderivatives of  $f(y)$  and  $g(x)$ , respectively, then this is the solution

$$\left[ F(y) = G(x) + C \right]$$

This equation defines  $y$  implicitly as a function of  $x$ . Sometimes you can use algebra to explicitly solve for  $y$ . The constant  $C$  can be adjusted to solve initial value problems.

#### Why the method works:

The use of differentials is disguising an application of the chain rule. Here is the explanation for the magic method: The differential equation

$$\left[ \frac{dy}{dx} = \frac{g(x)}{f(y)} \right]$$

CAN be rewritten as

$$\left[ \text{deqtn2} := f(y) \left[ \frac{dy}{dx} \right] = g(x) \right]$$

If  $y(x)$  is any solution to  $\text{deqtn2}$ , then the left side, namely

$$\left[ f(y(x)) \left[ \frac{dy}{dx} \right] \right]$$

is the derivative with respect to  $x$  of

$$\left[ F(y(x)) \right]$$

whenever  $F(y)$  is an antiderivative of  $f(y)$  (with respect to  $y$ ). This is just the chain rule! Thus if  $G(x)$  is any antiderivative of  $g(x)$  (w.r.t. $x$ ), we can legally antidifferentiate  $\text{deqtn2}$  with respect to  $x$  (on both sides) to get

$$\left[ F(y(x)) = G(x) + C \right]$$

which is what we got by magic before!

Example 1 page 31: We wish to solve

$$\frac{dy}{dx} = -6xy$$
$$y(0) = 7$$

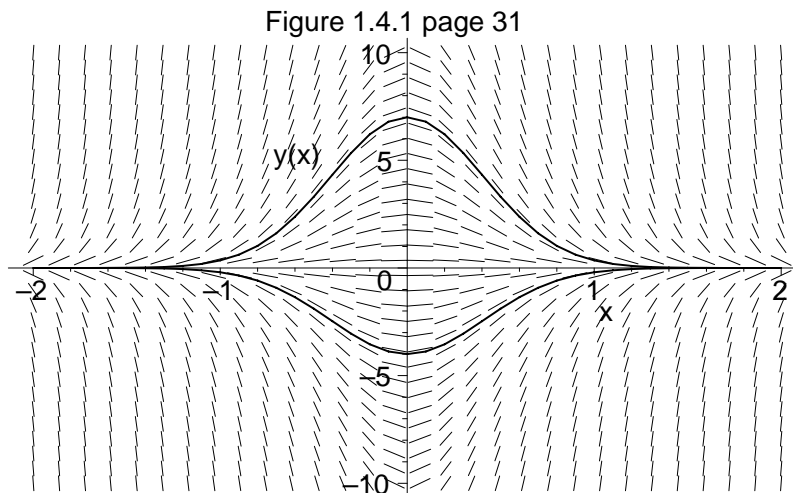
Work:

Notice, our method for the general solution doesn't actually give us the solution  $y(x)=0$ . Solutions which exist to separable DE's which are in addition to the ones we get are called "**singular solutions.**"

**slope field picture:**

```
[ > restart:with(plots):with(DEtools):  
> deqtn:=diff(y(x),x)=-6*x*y(x): #this is example 2  
dsolve({deqtn,y(0)=7},y(x)); #Maple solution  
DEplot(deqtn,y(x),x=-2..2,[y(0)=7],[y(0)=-4],y=-10..10,arrows=li  
ne, color=black,linecolor=black,dirgrid=[30,30],stepsize=.1,  
title='Figure 1.4.1 page 31'); #slope field with two solution  
graphs
```

$$y(x) = 7 e^{(-3x^2)}$$



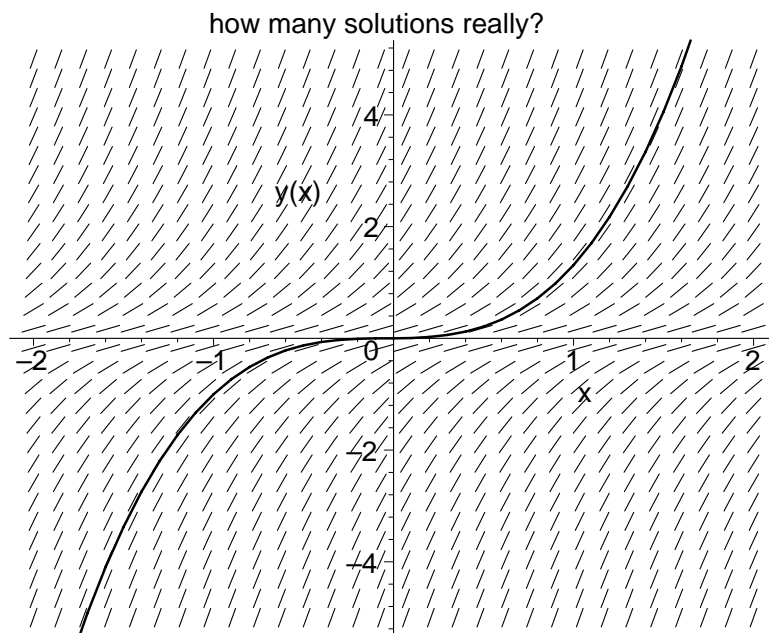
**Example extra:** (#33 page 27)

$$\frac{dy}{dx} = 3 y^{(2/3)}$$

$$y(-1) = -1$$

Solution: (There is a twist to this problem that will let us discuss the existence-uniqueness theorem.)

```
> deqtn:=diff(y(x),x)=3*abs(y(x))^(2/3.0)*sign(y(x)):
DEplot(deqtn,y(x),x=-2..2,{[y(-1)=-1]},y=-5..5,arrows=line,
color=black,linecolor=black,dirgrid=[30,30],stepsize=.1,
title='how many solutions really?'); #Maple missed a few
```



Example 3 page 32

```
> dy/dx=(4-2*x)/(3*y^2-5);  
y(1)=3;
```

$$\frac{dy}{dx} = \frac{4-2x}{3y^2-5}$$
$$y(1)=3$$

Work:

```
> deqtn:=diff(y(x),x)=(4-2*x)/(3*y(x)^2-5):  
part1:=DEplot(deqtn,y(x),x=-5..7,y=-5..5,{[y(1)=3]},arrows=line,  
color=black,linecolor=black,dirgrid=[40,40],stepsize=.1,  
title='Part of Figure 1.4.2 page 32 `):  
with(plots):  
part2:=implicitplot(y^3-5*y=4*x-x^2+9,x=-5..7,y=-5..5,color=black)  
:  
display({part1,part2});  
>
```

