Math 2250-3
PRACTICE FINAL EXAM
SOLUTIONS
Dec 9, 2000

   This exam is closed-book and closed-note. You may use a scientific calculator, but not one which is capable of graphing or of solving differential or linear algebra equations. **In order to receive full or partial credit on any problem, you must show all of your work** and **justify your conclusions.** This exam counts for 30% of your course grade. It has been written so that there are 200 points possible, however, and the point values for each problem are indicated in the right-hand margin. **Good Luck!**

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> restart:with(plots):with(DEtools):with(inttrans):with(linalg):
Warning, the name changecoords has been redefined

Warning, the names adjoint and hilbert have been redefined

Warning, the protected names norm and trace have been redefined and unprotected
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1) Consider the initial value problem

$$\frac{dy}{dx} + 3\,y = 2\,x$$

$$y(0) = 1$$

1a) Solve this problem using Chapter 1 techniques. By the way, there will be an integral table and a Laplace transform table at the end of the real test if you need it.

(10 points)

Use integrating factor, see section 1.5

$$\mathbf{e}^{(3\,x)}\left(\left(\frac{\partial}{\partial x}\,y(x)\right) + 3\,y(x)\right) = 2\,\mathbf{e}^{(3\,x)}\,x$$

Now integrate (by parts or integral #46 from table, after substituting u=3x)

$$\mathbf{e}^{(3\,x)}\,y(x) = \frac{2}{3}\,\mathbf{e}^{(3\,x)}\,x - \frac{2}{9}\,\mathbf{e}^{(3\,x)} + C$$

Solve for y(x):

$$y(x) = \frac{2}{3}\,x - \frac{2}{9} + C\,\mathbf{e}^{(-3\,x)}$$

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plug in initial conditions to find C:

$$1 = -\frac{2}{9} + C$$

$$C = \frac{11}{9}$$

$$y(x) = \frac{2}{3}x - \frac{2}{9} + \frac{11}{9}\, e^{(-3\,x)}$$

1b) Since the differential equation we are considering in this problem is a constant-coefficient linear one, the methods of Chapter 5 also apply. Explain how you would go about finding the general solution to the differential equation if you were using those techniques. In particular, how would you find the general solution to the homogeneous equation (and what is it)? What type of particular solution would you try to find?

(10 points)

The general solution will be a particular solution plus the general solution to the homogeneous problem. To solve the homogeneous problem we try solutions of the form

$$y = e^{(r\,x)}$$

which leads to the characteristic equation

$$r + 3 = 0$$

So we deduce the homogeneous solution is

$$yh := C\, e^{(-3\,x)}$$

We use the method of undetermined coefficients ("guess") and try for a particular solution of the form

$$yp = A\,x + B$$

If we substituted this into the differential equation and solved for A and B we would get A=2/3, B=-2/9, so that our solution would have the form

$$y(x) = \frac{2}{3}x - \frac{2}{9} + C\, e^{(-3\,x)}$$

We would then use the initial condition y(0)=1 to deduce C=11/9.

2) Consider the differential equation

$$\frac{dP}{dt} = -P^2 + 2\,P$$

which models a certain logistic population problem.
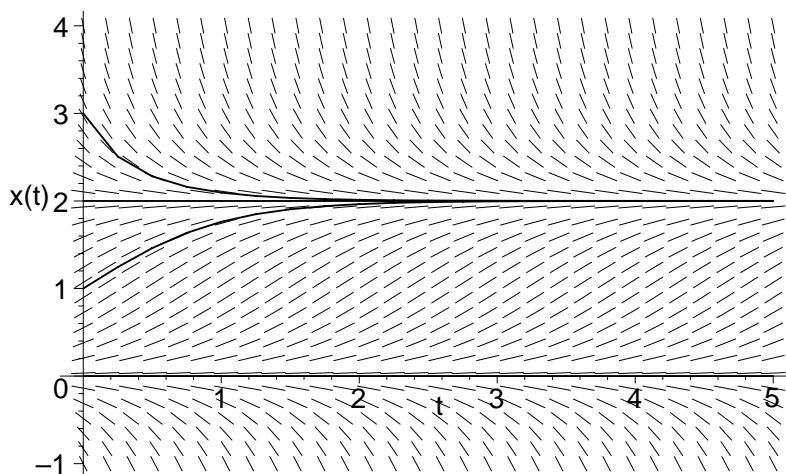
2a) Find the equilibrium solutions.

(4 points)

These are constant solutions, so dP/dt=0, so -P(P-2)=0, so P=0 or P=2.

2b) Sketch the slope field for this differential equation. Onto the slope field sketch graphs of the solutions to the four initial value problems with P(0)=0, P(0)=1, P(0)=2, P(0)=3 (You don't need formulas for the solutions to make the sketches!)

(7 points)

I'll use Maple. You would use the fact that the isoclines are horizontal lines, so that when P=1 the slope is -1+2=1, for example. You would plot slopes along several horizontal lines, and then starting at the

appropriate initial points you would sketch in the graphs of the solutions to the IVP's by following the slope field.

2c) Which of the equilibrium solutions are stable? Which are unstable?

(4 points)

P=0 is unstable (solutions starting near P=0 don't stay close to it), P=2 is stable(solutions starting close to P=2 stay close to it).

2d) Find an explicit solution to the initial value problem for this differential equation, with P(0)=1. Verify that your limiting population agrees with what your sketch predicted in part 1b).

(15 points)

This is a separable DE:

$$\frac{dP}{P(P-2)} = -dt$$

do partial fractions:

$$\frac{1}{2}\left(\frac{1}{P-2} - \frac{1}{P}\right)dP = -dt$$

integrate:

$$\ln\left(\left|\frac{P-2}{P}\right|\right) = -2\,t + C$$

Find C from initial value:

$$0 = C$$

exponentiate:

$$\left|\frac{P-2}{P}\right| = \mathbf{e}^{(-2\,t)}$$

Use initial value to decide on the plus or minus from absolute value

$$\frac{2-P}{P} = \mathbf{e}^{(-2\,t)}$$

$$\begin{bmatrix} 2 - P = P\,\mathbf{e}^{(-2\,t)} \\[6pt] 2 = P\,(1 + \mathbf{e}^{(-2\,t)}) \\[6pt] P = 2\,\dfrac{1}{1 + \mathbf{e}^{(-2\,t)}} \end{bmatrix}$$

It is easy to check that P(0)=1, and that as t-> infinity, exp(-2t)->0, so P->2 as predicted by the slope field picture.

3) Consider the homogeneous differential equation

$$\frac{d^2 x}{dt^2} + 8\,\frac{dx}{dt} + 20\,x = 0$$

3a) If this was modeling a mass-spring configuration like we studied in Chapter 5 of Edwards-Penney, and if the mass was 3 kg, what values of coefficient of friction and spring constant would lead to the differential equation above? (1 point for getting the units correct, 2 points for the correct numerical values).

(6 points)

Since mass is coefficient of acceleration, we recover the original model equation by multiplying the given one by 3. The original coefficient of friction is 24 kg/sec, and original spring constant is 60 newtons/meter.

3b) What kind of damping is exhibited by this mass-spring system?

(4 points)

The characteristic equation is

$$\begin{bmatrix} r^2 + 8\,r + 20 = 0 \end{bmatrix}$$

which has complex roots

$$\begin{bmatrix} \text{-4} + 2\,I,\ \text{-4} - 2\,I \end{bmatrix}$$

So the system is underdamped.

3c) Solve the initial value problem for the differential equation above, where x(0)=5 and dx/dt(0)=4. Use the methods of Chapter 3.

(20 points)

From the characteristic roots and Euler's equation we know that the general solution is

$$\begin{bmatrix} x(t) := \mathbf{e}^{(-4\,t)}\,(c1\,\cos(2\,t) + c2\,\sin(2\,t)) \end{bmatrix}$$

and the derivative function is

$$\begin{bmatrix} -4\,\mathbf{e}^{(-4\,t)}\,(c1\,\cos(2\,t) + c2\,\sin(2\,t)) + \mathbf{e}^{(-4\,t)}\,(-2\,c1\,\sin(2\,t) + 2\,c2\,\cos(2\,t)) \end{bmatrix}$$

So if we substitute in initial values we get two equations for c1 and c2:

$$\begin{bmatrix} c1 = 5 \\[4pt] -4\,c1 + 2\,c2 = 4 \\[4pt] c1 = 5 \\[4pt] c2 = 12 \\[4pt] x(t) = \mathbf{e}^{(-4\,t)}\,(5\,\cos(2\,t) + 12\,\sin(2\,t)) \end{bmatrix}$$

3d) Re-solve the initial value problem of 3c), this time using the Laplace Transform techniques of Chapter 10. Of course, your answers to 3c) and 3d) should agree if you do both parts correctly.

(20 points)

$$s^2 \, X(s) - 5 \, s - 44 + 8 \, s \, X(s) + 20 \, X(s) = 0$$

$$X(s) \, (s^2 + 8 \, s + 20) = 5 \, s + 44$$

$$X(s) = \frac{5 \, s + 44}{s^2 + 8 \, s + 20}$$

complete the square (and then complete the linear in the numerator)

$$X(s) = \frac{5 \, s + 44}{(s+4)^2 + 4}$$

$$X(s) = 5 \, \frac{s+4}{(s+4)^2 + 4} + \frac{24}{(s+4)^2 + 4}$$

Now use tranlation theorem and Laplace table to find x(t):

$$x(t) = 5 \, e^{(-4\,t)} \cos(2\,t) + 12 \, e^{(-4\,t)} \sin(2\,t)$$

4) Consider the following two-tank configuration. In tank one there is uniformly mixed volume of V1 gallons, and pounds of solute x(t). In tank two there is mixed volume of V2 gallons and pounds of solute y(t). Water us pumped into tank one at a constant rate of r1 gallons/minute from an outside source, and this water has a constant solute concentration of c1 pounds/gallon. Water is pumped from tank one to tank two at constant rate of r2 gallons/minute, from tank two to tank one at constant rate r3 gallons/minute, and out of the tank system at constant rate r4 gallons/minute.

I don't know how to draw pictures in Maple. (A very painful thing would be to display a complicated 2-d plot, maybe there are ways to import pictures, I don't know.) So picture tank1 to the left of tank2, pipes running between them, an additional input pipe for tank1 and an output tank from tank2, see figure page 386.

4a) What conditions on the rates r1,r2,r3,r4 are necessary to guarantee that the volumes V1 and V2 remain constant in time?

(5 points)

for V1 to remain constant, we need r1+r3-r2 = 0. For V2 to remain constant we need r2-r4=0

4b) Write the system of first order differential equations which governs the process described above. Do not try to solve these DE's.

(10 points)

$$\frac{\partial}{\partial t} \, x(t) = r1 \, c1 - \frac{r2 \, x}{V1} + \frac{r3 \, y}{V2}$$

$$\frac{\partial}{\partial t} \, y(t) = \frac{r2 \, x}{V1} - \frac{(r3 + r4) \, y}{V2}$$

5) Let A be the matrix:

$$A := \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

5a) Compute the determinant of A and use it to determine whether A is singular or nonsingular.

(5 points)

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$$\mathrm{Det}(A) = -1$$

so A is non-singular.

5b) Does your answer to (5a) let you deduce the reduced row echelon form of A? Explain.

(5 points)

rref(A) is the identity matrix, since that is the rref of any non-singular matrix

5c) Find the inverse matrix to A. (You may use the next page.)

(15 points)

Method I is to augment A with the identity matrix and put the augmented matrix into rref, so that it is the identity matrix augmented with the inverse of A:

$$AaugI := \begin{bmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

> rref(AaugI);

$$\begin{bmatrix} 1 & 0 & 0 & -1 & -1 & 1 \\ 0 & 1 & 0 & -2 & -1 & 1 \\ 0 & 0 & 1 & 4 & 3 & -2 \end{bmatrix}$$

$$Ainv := \begin{bmatrix} -1 & -1 & 1 \\ -2 & -1 & 1 \\ 4 & 3 & -2 \end{bmatrix}$$

Method 2 is to use the adjoint formula. First compute the cofactor matrix:

> cof(A):=matrix(3,3,[1,2,-4,1,1,-3,-1,-1,2]);

$$\mathrm{cof}(A) := \begin{bmatrix} 1 & 2 & -4 \\ 1 & 1 & -3 \\ -1 & -1 & 2 \end{bmatrix}$$

take its transpose to get the adjoint

> adj(A):=transpose(cof(A));

$$\text{adj}(A) := \begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & -1 \\ -4 & -3 & 2 \end{bmatrix}$$

divide the adjoint by det(A) to get inverse(A)

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> invA:=1/det(A)*adj(A);
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$$invA := -\begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & -1 \\ -4 & -3 & 2 \end{bmatrix}$$

6) Let B be the matrix

$$B := \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -1 & 1 & 4 \end{bmatrix}$$

6a) Find a basis for R^3 made out of eigenvectors of B

(20 points)

first find the eigenvalues (whcih for a triangular matrix will be the diagonal entries):

$$\text{Det}(B - I\lambda) = (1 - \lambda)(2 - \lambda)(4 - \lambda)$$

For each eigenvalue find an eigenspace basis. You would do this by computing rref(B-lambd*I) for each lambda. I will use Maple:

```
> eigenvects(B);
```

$$\left[1, 1, \left\{\left[\frac{3}{2}, \frac{-3}{2}, 1\right]\right\}\right], [2, 1, \{[0, -2, 1]\}], [4, 1, \{[0, 0, 1]\}]$$

So we may take our basis to be

$$\left\{ \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

6b) Use your answer from part 6a) to write down the general solution to the system d**x**/dt = B**x**, of three first order differential equations.

(5 points)

$$\mathbf{x}(t) = c1\, \mathbf{e}^t \begin{bmatrix} 3 \\ -3 \\ 2 \end{bmatrix} + c2\, \mathbf{e}^{(2t)} \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} + c3\, \mathbf{e}^{(4t)} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

7) Consider the following configuration of springs, with positive displacements from equilibrium

measured to the right, as indicated.

This configuration has the wall on the RIGHT, like a reflected version of the picture on page 386. Looking from left to right, we see a mass m1, connected to a spring with constant k1, connected to a mass m2, connected to a spring with constant k2, connected to the wall. Displacement of mass m1 to the right of equilibrium is called x(t), displacement to the right for mass m2 is called y(t).

7a) Derive the system of second order differential equations which models this system. Assume that there are no external forces.

(5 points)

$$m1\left(\frac{\partial^2}{\partial t^2}x(t)\right) = k1\ (y - x)$$

$$m2\left(\frac{\partial^2}{\partial t^2}y(t)\right) = -k1\ (y - x) - k2\ y$$

7b) Assume that in appropriate units m1=2, m2=2, k1=4, k2=6. Show that in this case your system above reduces to

$$\left[\begin{array}{c}\dfrac{d^2 x}{dt^2}\\[2mm]\dfrac{d^2 y}{dt^2}\end{array}\right] = \left[\begin{array}{c}-2\,x + 2\,y\\[1mm]2\,x - 5\,y\end{array}\right]$$

(5 points)

This is easy to see since k1/m1=2, k1/m2=2, k2/m2=3.

7c) Find the general solution to the unforced system (7b).

(15 points)

For A defined as

$$A := \left[\begin{array}{cc}-2 & 2\\ 2 & -5\end{array}\right]$$

we find the eigenvalues and eigenvectors. The square roots of the opposites of the eigenvalues are the fundamental angular frequencies, the eigenvectors are the fundamental modes.
```
> eigenvects(A);
```
$$[-1,\ 1,\ \{[2,\ 1]\}],\ [-6,\ 1,\ \{[1,\ -2]\}]$$

We deduce that the general solution is

$$x1(t) = (c1\ \cos(t) + c2\ \sin(t))\left[\begin{array}{c}2\\1\end{array}\right] + (c3\ \cos(\sqrt{6}\ t) + c4\ \sin(\sqrt{6}\ t))\left[\begin{array}{c}1\\-2\end{array}\right]$$

7d) Assuming omega is not a natural frequence for the problem above, find a particular solution to the

forced system

$$\begin{bmatrix} \dfrac{d^2 x}{dt^2} \\[2em] \dfrac{d^2 y}{dt^2} \end{bmatrix} = \begin{bmatrix} -2\,x + 2\,y + \cos(\omega\, t) \\[1em] 2\,x - 5\,y - \cos(\omega\, t) \end{bmatrix}$$

(10 points)

We try a particular solution of the form xp=cos(wt)c, where we find c by substituting xp(t) into the inhomogeneous DE.   We get the equation

$$c := \begin{bmatrix} c1 \\ c2 \end{bmatrix}$$

$$b := \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$-\omega^2 \cos(\omega\, t)\, c = \cos(\omega\, t)\, A\, c + \cos(\omega\, t)\, b$$

We divide by the scalar function cos(wt), and reduce to

$$-b = (A + I\,\omega^2)\, c$$

i.e.

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 + \omega^2 & 2 \\ 2 & -5 + \omega^2 \end{bmatrix} \begin{bmatrix} c1 \\ c2 \end{bmatrix}$$

Which we can solve with Cramer's rule or via an inverse matrix, yielding:

$$\begin{bmatrix} c1 \\ c2 \end{bmatrix} = \begin{bmatrix} \dfrac{3 - \omega^2}{(\omega^2 - 6)(\omega^2 - 1)} \\[2em] \dfrac{\omega^2}{(\omega^2 - 6)(\omega^2 - 1)} \end{bmatrix}$$