Name...
I.D. number...

Math 2210-4
FINAL EXAM
May 2, 2005

This exam is closed-book. You are allowed a 4" by 6" index card of notes and formulas. You may also use a scientific calculator, but not one which is capable of doing integration or solving equations. Integral tables are included with this exam. **In order to receive full or partial credit on any problem, you must show all of your work** and **justify your conclusions.** This exam counts for 30% of your course grade. It has been written so that there are 150 points possible, and the point values for each problem are indicated in the right-hand margin. **Good Luck!**

1a) Compute

$$\int_{-1}^{1}\int_{x^2}^{1} y\,dy\,dx$$

(5 points)

1b) Sketch the region of integration for the iterated integral in (1a). Label the boundary curves and the points at which they intersect.

(5 points)

1c) Compute the double integral in (1a) over the same regions, but with the order of integration reversed.

(7 points)

1d) Suppose that a laminate in the plane region you sketched for (1b) has density function $\delta(x, y) = y$, so that your integral computations in 1a) and 1c) were for the mass of the laminate. Find the center of mass of this laminate. (Hint: You may use symmetry to deduce one of the coordinates.)

(8 points)

2) Consider the parametric curve with position vector given by

$$\mathbf{r}(\mathrm{t}) = \langle 1 - t\,,\, t^2 - 2\,t \rangle.$$

There is a sketch of part of this curve below.

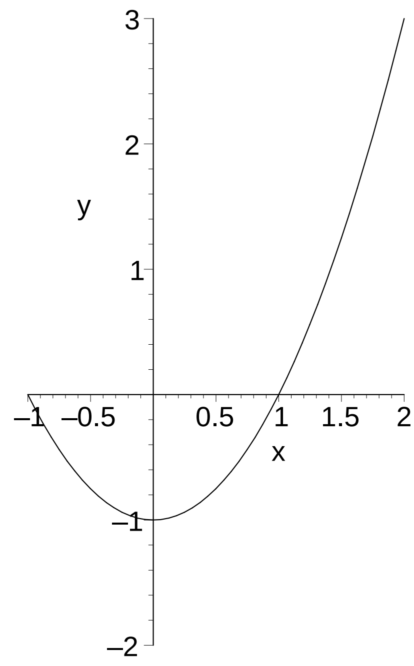
2a) Show that (the range of) this curve lies on the parabola with equation

$$y = x^2 - 1$$

(4 points)

2b) Find $\mathbf{r}(0)$, $\mathbf{r}'(0)$, and $\mathbf{r}''(0)$. Find the unit tangent and normal vectors, **T** and **N**, when t = 0. Carefully indicate the point with position **r**(0) in the picture below, and add the remaining four vectors to the picture, making sure to draw them as accurately as possible using a piece of paper or index card ruler.

(14 points)



2c) Use the graph on the previous page, an appropriate right triangle which you add to the picture, and an index card ruler to measure and record below (accuracy within 0.2 suffices) the components of the acceleration $\mathbf{r}''(0)$ in the tangential and normal directions. indicate which sides of the triangle correspond to which components.

(6 points)

2d) Find the exact values of the acceleration components in the tangential (T) and normal (N) directions, when t = 0. You can either use the "Roller coaster equation", or the dot product to compute these components! (Your values should be close to the numerical approximations in (2d).)

(6 points)

3) Consider the function

$$f(x, y) = 2x^2 + 3xy - y^2$$

3a) Compute the gradient of f(x,y), and its value at the point $(x, y) = (1, 2)$.

(4 points)

3b) Find a vector *tangent* to the level curve f(x,y)=4, at the point (1,2).

(4 points)

3c) Use differentials to approximate $f(1.02, 1.97)$, using $f(1, 2) = 4$. (Hint: the exact value is 4.2281.)

(6 points)

3d) A particle is moving in the x-y plane with some position vector **r**(t). Suppose that **r**(0) = (1, 2), and that **r**′(0) = < 3, -1 >. What is the derivative of the composition f(**r**(t)), at t = 0 ?

(6 points)

4) Consider the vector field F defined in the half of the x-y plane with x>0, and with formula given by

$$F(x, y) = < M(x, y), N(x, y) > \; = \; < 2y - \frac{y}{x^2} \;,\; \frac{1}{x} + 2x + 3 > .$$

4a) Do a computation which verifies that <M, N > must be a gradient vector field (for x>0) .

(4 points)

4b) Find a function f(x,y) so that its gradient is F.

(8 points)

4c) Let C be the segment of the line $y = 2\,x$, from the point (1,2) to the point (2,4). Find the value of

$$\int_C M\,dx + N\,dy$$

by explicitly computing the line integral.

(7 points)

4d) Recompute the value of the line integral in (4c), this time using the function f(x,y) you found (4b).

(6 points)

5) What are the optimal dimensions of an open-topped rectangular box, so that the total volume is a fixed value V,

$$x\,y\,z = V$$

and so that the total surface area (of the four sides and the bottom) is minimized? You may solve this problem either of the two ways we discussed in this course.

(10 points)

6) Consider the (ice-cream cone like) region which is the intersection of the points inside the radius-3 sphere and inside (above) the cone $z = \sqrt{x^2 + y^2}$. Precisely the points in our region satisfy both the inequalities

$$x^2 + y^2 + z^2 \leq 9$$
$$\sqrt{x^2 + y^2} \leq z$$

Find the volume of this region. (Hint: try spherical or cylindrical coordinates!)

(10 points)

7) Consider the disk D of radius 2, defined by the inequality

$$x^2 + y^2 \leq 4$$

with its boundary curve C given by the circle

$$x^2 + y^2 = 4$$

Consider the vector field

$$F = \langle M(x,y),N(x,y)\rangle = \langle x-y,\ y+x \rangle.$$

7a) Compute the flux integral

$$\int_C F \quad n\, ds,$$

directly, where n is the unit exterior normal to C. You can complete this computation by parameterizing the circle and using the general formula for flux integrals. Alternately, you might notice that that F n turns out to be constant along the circle and so deduce the value of the integral from geometry.

(8 points)

7b) Check your work in (7a) by using the divergence theorem to deduce the value of the flux integral by computing an appropriate double integral over the disk D. (Note, you may evaluate this resulting integral using geometry.)

(7 points)

8) Consider the triangle with vertices (4,0,0), (0,3,0) and (0,0,12). This triangle is the intersection of the plane

$$3x+4y+z=12$$

with the first octant.

8a) Find the area of this triangular surface.

(5 points)

8b) Consider the vector field F = <-y, x, z >. Compute the curl of F.

(5 points)

8c) Let C be the boundary of the triangle defined above, traversed in the counterclockwise direction as veiwed from above. Use Stoke's theorem to find the value of the line integral

$$\int_C F \quad T\,ds.$$

(5 points)