

Name..... SOLUTIONS
I.D. number.....

Math 2210-4

Exam 1

February 16, 2005

The exam is closed-book and closed-note. You will be allowed the use of a scientific calculator (only), but not one which is capable of graphing or of solving linear algebra equations. You won't actually need the calculator, but there could be times when numerical values would be a hint about whether you've correctly solved a problem. **In order to receive full or partial credit on any problem, you must show all of your work and justify your conclusions.** There are 100 points possible. The point values for each problem are indicated in the right-hand margin. **Good Luck!**

1) Consider the points $P = (1, 2, -1)$ and $Q = (-1, 3, 0)$, and the vector $\vec{v} = \langle -1, 3, 2 \rangle$.

1a) Compute the cross product $\vec{PQ} \times \vec{v}$. Be careful, because you will be using your answer in several other parts of problem 1.

(10 points)

$$\vec{PQ} = \langle -2, 1, 1 \rangle$$
$$\vec{PQ} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 1 & 1 \\ -1 & 3 & 2 \end{vmatrix} = \langle \begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix}, -\begin{vmatrix} -2 & 1 \\ -1 & 2 \end{vmatrix}, \begin{vmatrix} -2 & 1 \\ -1 & 3 \end{vmatrix} \rangle$$
$$= \langle -1, 3, -5 \rangle$$

1b) Do computations which verify that your answer to (1a) is perpendicular to both \vec{PQ} and to \vec{v} .

(5 points)

$$\langle -1, 3, -5 \rangle \cdot \langle -2, 1, 1 \rangle = 2 + 3 - 5 = 0 \quad \text{so } \vec{PQ} \perp (\vec{PQ} \times \vec{v})$$

$$\langle -1, 3, -5 \rangle \cdot \langle -1, 3, 2 \rangle = 1 + 9 - 10 = 0 \quad \text{so } \vec{v} \perp (\vec{PQ} \times \vec{v})$$

1c) Let L be the line through P with direction vector \vec{v} . Express this line parametrically.

(5 points)

position vector $\vec{r}(t) = \vec{OP} + t\vec{v}$

$$\vec{r}(t) = \langle 1, 2, -1 \rangle + t \langle -1, 3, 2 \rangle$$

(or, could write

$$\begin{aligned} x &= 1 - t \\ y &= 2 + 3t \\ z &= -1 + 2t \end{aligned}$$

1d) Find an equation for the plane which contains the line L of (1c), and the point Q. Note that you've already done part of the work for this problem, in (1a).

plane normal \perp to \vec{PQ} & \vec{v} , so e.g. $\vec{PQ} \times \vec{v} = \langle -1, 3, -5 \rangle$ (5 points)
 $\parallel \langle 1, -3, 5 \rangle$

$$1 \cdot (x-1) - 3(y-2) + 5(z+1) = 0$$

$$x - 3y + 5z = 1 + 6 + 5 = 0$$

$$\boxed{x - 3y + 5z = -10}$$

1e) A formula for the (nearest) distance from a point Q to a line L is

$$d = \frac{|\vec{PQ} \times \vec{v}|}{|\vec{v}|}$$

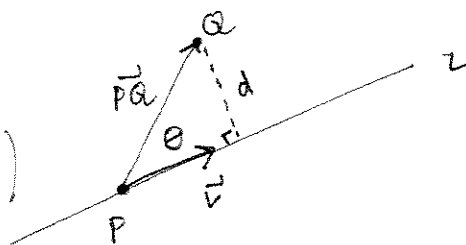
where P is any point on the line L and \vec{v} is a direction vector for it. Use this formula to find the distance from $Q = (1, 3, 0)$ to the line L in (1c). Note that you've done part of the work for this computation already, in (1a).

$$d = \frac{|\langle 1, -3, 5 \rangle|}{|\langle -1, 3, 2 \rangle|} = \frac{\sqrt{1+9+25}}{\sqrt{1+9+4}} = \frac{\sqrt{35}}{\sqrt{14}} = \boxed{\frac{\sqrt{5}}{\sqrt{2}}} \quad (5 \text{ points})$$

1f) Using your knowledge of the cross product geometry, and the figure drawn below, derive the formula for distance from a point to a line which was given to you in (1d).

(5 points)

$$\begin{aligned} d &= |\vec{PQ}| \sin \theta \quad (\text{geometry}) \\ |\vec{PQ} \times \vec{v}| &= |\vec{PQ}| |\vec{v}| \sin \theta \quad (\text{cross prod property}) \\ &= |\vec{v}| d \\ \text{So } d &= \frac{|\vec{PQ} \times \vec{v}|}{|\vec{v}|} \end{aligned}$$



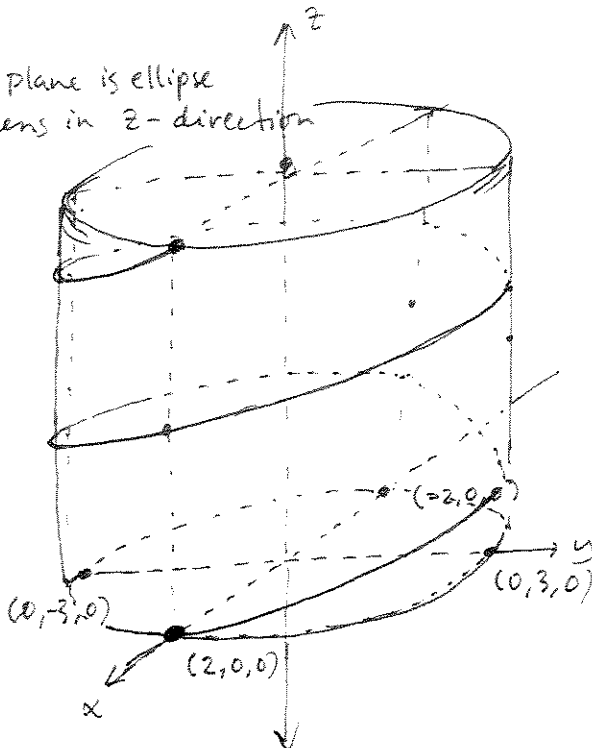
2) Consider the cylinder which satisfies the equation

$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

2a) Sketch the portion of this cylinder for which $0 \leq z \leq 4$, as much to scale as possible. Label all axis intercepts.

(10 points)

trace in x-y plane is ellipse
cylinder opens in z-direction



2b) Consider the helix with position vector

$$\vec{r}(t) = \langle 2 \cos(\pi t), 3 \sin(\pi t), t \rangle.$$

Do a computation which verifies that this curve lies on the cylinder we are considering.

(5 points)

$$\text{if } \langle x, y, z \rangle = \langle 2 \cos \pi t, 3 \sin \pi t, t \rangle$$

$$\text{then } \frac{x^2}{4} + \frac{y^2}{9} = \frac{4 \cos^2 \pi t}{4} + \frac{9 \sin^2 \pi t}{9} = \cos^2 \pi t + \sin^2 \pi t = 1$$

so point satisfies cylinder eqn.

2c) Add the portion of the curve in (2b) for which $0 \leq t \leq 4$, onto your sketch of the cylinder in (2a).

(5 points)

as $0 \leq t \leq 4$ the projected ~~curve~~ ^{ellipse} to x-y plane

$\langle 2 \cos \pi t, 3 \sin \pi t \rangle$ has $0 \leq \pi t \leq 4\pi$,

so goes around ellipse twice.

3) Consider the parametric curve with position vector given by

$$\vec{r}(t) = \langle 3t, t^3 \rangle.$$

There is a sketch of part of the curve on the next page.

3a) Show that this curve lies on the graph of the cubic equation

$$y = \frac{x^3}{27}$$

$$y = t^3$$

$$\frac{x^3}{27} = \frac{(3t)^3}{27} = t^3.$$

$$\text{So } y = \frac{x^3}{27}$$

(4 points)

3b) Compute $\vec{r}'(t)$, $\vec{r}''(t)$, and the speed $v(t)$.

$$\vec{r}'(t) = \langle 3, 3t^2 \rangle$$

$$v(t) = |\vec{r}'(t)| = 3\sqrt{1+t^4}$$

$$\vec{r}''(t) = \langle 0, 6t \rangle$$

(6 points)

3c) Using your favorite formula, find the curvature of this curve at the point on the curve corresponding to $t = -1$.

$$\kappa = \frac{|x'y'' - x''y'|}{v^3}$$

(7 points)

$$\vec{r}'(-1) = \langle 3, 3 \rangle$$

$$\vec{r}''(-1) = \langle 0, -6 \rangle$$

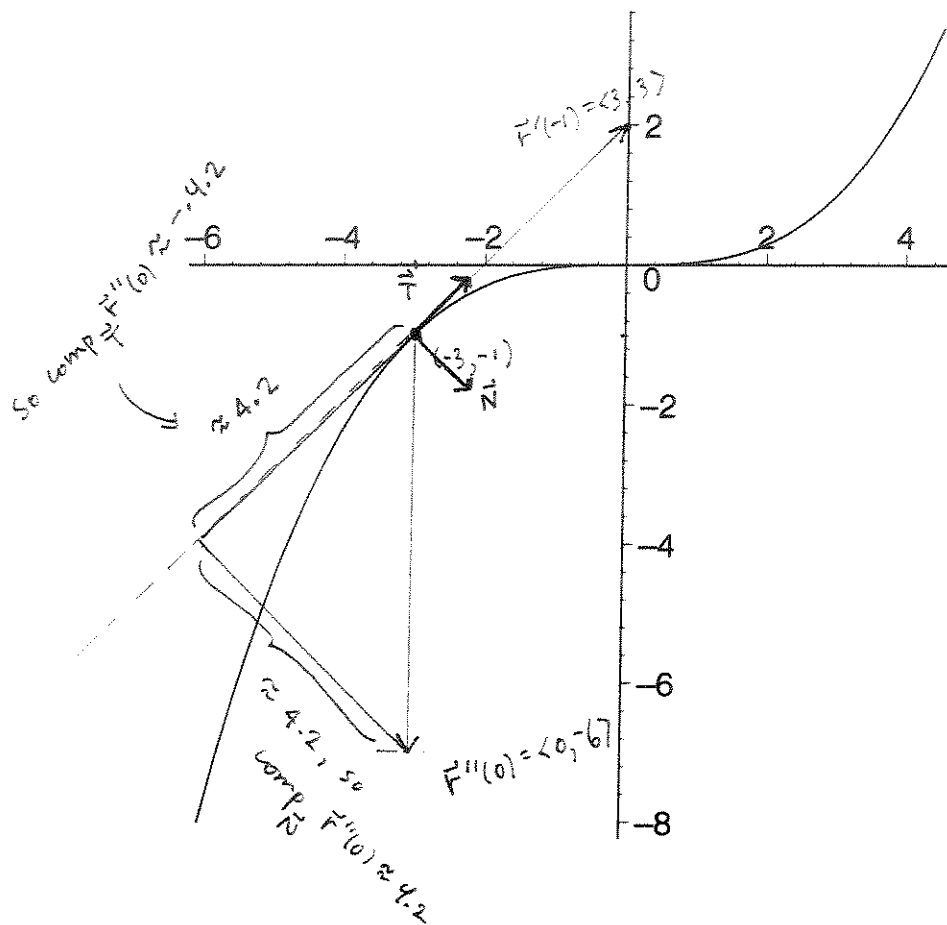
$$v(-1) = 3\sqrt{2}$$

$$\text{so } \kappa(-1) = \frac{|3(-6) - 0|}{(3\sqrt{2})^3} = \frac{18}{54\sqrt{2}} = \frac{1}{3\sqrt{2}}$$

$$= \frac{1}{3\sqrt{2}}$$

$$\approx 0.24$$

Here is a picture of a piece of our curve, with position vector $\vec{r}(t) = \langle 3t, t^3 \rangle$:



3d) Label the point with position vector $\vec{r}(-1)$ into the picture above. Compute, and accurately draw the vectors $\vec{r}'(-1)$ and $\vec{r}''(-1)$ into the picture, in the appropriate location(s), using an index card ruler. Also, compute and draw in the unit tangent and normal vectors, \vec{T} and \vec{N} , when $t = -1$. (Hint: you can find \vec{N} easily because it is perpendicular to \vec{T} and the curve lies in a plane.)

(10 points)

$$\vec{r}(-1) = \langle -3, -1 \rangle$$

$$\vec{r}'(-1) = \langle 3, 3 \rangle$$

$$\vec{r}''(-1) = \langle 0, -6 \rangle$$

~~Not this~~

$$\text{so } \vec{T} = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \frac{1}{\sqrt{2}} \langle 1, 1 \rangle$$

$$\text{so } \vec{N} = \frac{1}{\sqrt{2}} \langle 1, -1 \rangle$$

3e) Use the graph on the previous page, an appropriate right triangle which you add to the picture, and an index card ruler to measure and record (accuracy within 0.4 suffices) the components of the acceleration $\vec{r}''(0)$ in the tangential and normal directions. indicate which sides of the triangle correspond to which components.

(6 points)

$$\text{See previous page} \rightarrow \text{comp}_{\vec{T}} \vec{r}''(0) \approx -4.2$$

$$\text{comp}_{\vec{N}} \vec{r}''(0) \approx 4.2$$

3f) Use the dot product to find the exact values of the acceleration components in the tangential (T) and normal (N) directions, when $t = -1$. (These should be close to your numerical approximations in (3e)!) (6 points)

$$\begin{aligned} \text{comp}_{\vec{T}} \vec{r}''(0) &= \vec{r}''(0) \cdot \vec{T} && \text{because } \vec{T} \text{ is a unit vector} \\ &= \langle 0, -6 \rangle \cdot \frac{1}{\sqrt{2}} \langle 1, 1 \rangle = \frac{-6}{\sqrt{2}} = -3\sqrt{2} \approx -4.24 \\ \text{comp}_{\vec{N}} \langle 0, -6 \rangle &= \langle 0, -6 \rangle \cdot \frac{1}{\sqrt{2}} \langle 1, -1 \rangle = \frac{6}{\sqrt{2}} \approx 4.24 \end{aligned}$$

3g) Recompute the components in (3f) by using the roller coaster equation

$$\vec{r}''(t) = \left(\frac{d}{dt} v(t) \right) \vec{T} + \kappa v^2 \vec{N}$$

where the scalar v represents speed and κ is the curvature. Note that you've already done most of the computations you need in (3b), (3c).

(6 points)

$$\begin{aligned} v(t) &= 3 \cdot (1+t^4)^{1/2} \\ v'(t) &= 3 \cdot \frac{1}{2} (1+t^4)^{-1/2} 4t^3 \end{aligned}$$

$$v'(-1) = \frac{3}{2} \cdot \frac{1}{\sqrt{2}} 4(-1) = -3\sqrt{2} = \text{comp}_{\vec{T}} \vec{r}''(-1)$$

$$\begin{aligned} v(1) &= 3\sqrt{2} \\ \kappa(-1) &= \frac{1}{3\sqrt{2}} \end{aligned}$$

$$\left. \begin{aligned} v(1) &= 3\sqrt{2} \\ \kappa(-1) &= \frac{1}{3\sqrt{2}} \end{aligned} \right\} \kappa v^2 = \frac{1}{3\sqrt{2}} (3\sqrt{2})^2 = 3\sqrt{2} = \text{comp}_{\vec{N}} \vec{r}''(0)$$

} agrees with (3f)