Math 1210-3
Friday March 5

Review!

f diffble at $\vec{p}_0$:

$$f(\vec{p}_0+\vec{h}) = f(\vec{p}_0) + \nabla f(\vec{p}_0)\cdot\vec{h} + \vec{h}\cdot\vec{\varepsilon}(h) \qquad \text{where } \vec{\varepsilon}(\vec{h})\to 0 \text{ as } \vec{h}\to 0$$

led to

$$D_{\vec{u}} f(\vec{p}_0) := \lim_{h\to 0} \frac{f(p_0+h\vec{u})-f(p_0)}{h} \qquad (|\vec{u}|=1) \qquad \text{Directional derivative.}$$

$$= \nabla f(\vec{p}_0)\cdot\vec{u}$$

$$\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t))\cdot\vec{r}\,'(t) \qquad \text{Chain rule}$$

equals speed $\|\vec{r}\,'(t)\|$ times direction! $\vec{u} = \dfrac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}$

- Notice how similar the chain rule formula is to the directional derivative formula, and that this is sensible. (Ben brought this up on Wednesday!)

<u>Example</u>: One fine spring day is SCL, the temperature on the East Bench is approximately

$$T(x,y) = 50 - x - y - .2xy - .5x^2 - \tfrac{1}{3}y^3 \quad \text{degrees F}$$

where (x,y) are EW/NS coords in miles from downtown.
At this instant a student is at $(2,0)$ and ~~walking~~ biking northeast at 4 mph. At what rate is the temperature of the student's surroundings changing?

<u>Example</u> Chain rule for partial derivs, page 4 Wednesday

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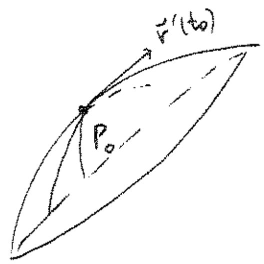
today we discuss consequences of, and topics related to the chain rule, §12.7

Theorem: Let $F(x,y,z)=k$ be the equation of a level surface in $\mathbb{R}^3$, for the function $F(x,y,z)$. Let (x_0,y_0,z_0) be on this level surface. Then $\nabla f(x_0,y_0,z_0)$ is a normal vector to the tangent plane to the surface at (x_0,y_0,z_0)



$F(x,y,z)=k$
(piece of surface

proof. Let $\vec{r}(t)$ be a curve lying on the level surface, passing thru $\vec{p}_0=(x_0,y_0,z_0)$ at time t_0

Then $F(\vec{r}(t)) \equiv k$

so $\frac{d}{dt}F(\vec{r}(t)) \equiv 0$

|| chain rule

$\nabla f(\vec{r}(t))\cdot \vec{r}(t)$

at t_0, $\nabla f(\vec{p}_0)\cdot \vec{r}'(t_0)=0$

$\Rightarrow \nabla f(\vec{p}_0) \perp \vec{r}'(t_0)$.

But this is true for all possible curves $\vec{r}(t)$ on M, passing thru p,

so $\nabla f(\vec{p}_0) \perp$ to all tangent vectors to M at p.

(Notice the idea of this proof also shows that for a level curve $f(x,y)=k$ at (x_0,y_0) on this curve, $\nabla f(x_0,y_0)$ is $\perp$ to the curve.)

Thus: The equation of the tangent plane at (x_0,y_0,z_0) is

$$\boxed{\begin{aligned} &\nabla F(x_0,y_0,z_0)\cdot\langle x-x_0,\, y-y_0,\, z-z_0\rangle = 0 \\ \text{i.e.}\quad &F_x(\vec{p}_0)(x-x_0)+F_y(\vec{p}_0)(y-y_0)+F_z(\vec{p}_0)(z-z_0)=0 \end{aligned}}$$

<u>Example</u>: Find the eqtn for the tangent plane to

$$\frac{x^2}{4} + \frac{y^2}{8} + \frac{z^2}{16} = 1$$

at $(1, 2, 2)$

(a): Solve for $z(x,y)$, use old formula. (Hard!)
i.e. $z = T(x,y)$

(b) Use old formula, but find z_x, z_y implicitly. (better)

(c) Use Result from last page (best).

Differential approximation

Using the tangent approximation $T(x)$ to $f(x)$ near x_0, $T(x,y)$ to $f(x,y)$ near (x_0,y_0), $T(x,y,z)$ to $f(x,y,z)$ near (x_0,y_0,z_0)

is also called differential approximation, and is often written using "differentials."

and we did some examples of these last Friday, §12.4 differentiability

<u>1210</u> : $y = f(x)$ You've seen this!

$$f(x+\Delta x) \approx f(x) + f'(x)\Delta x + \Delta x\, \varepsilon(x)$$

$$dx := \Delta x$$
$$\Delta y := f(x+\Delta x) - f(x)$$
$$\boxed{dy := f'(x)\,dx}$$ is the tangent function approx to Δy

<u>2210</u> $z = f(x,y)$

$$f(x+\Delta x, y+\Delta y) \approx f(x,y) + f_x(x,y)\Delta x + f_y(x,y)\Delta y + \Delta x\, \varepsilon_1(\Delta x, \Delta y) + \Delta y\, \varepsilon_2$$

$$dx := \Delta x$$
$$dy := \Delta y$$
$$\Delta z = \Delta f := f(x+\Delta x, y+\Delta y) - f(x,y)$$
$$\boxed{dz := f_x\,dx + f_y\,dy}$$ is the tangent approx to Δz

<u>etc.</u>

<u>Remark</u> : differentials are compatible with the chain rule. Notice that in both 1210 & 2210 if x (x,y) are fns of t, get chain rule by dividing either box by dt

<u>Example</u> : A steel cylinder is measured to be 10 cm tall, (±.1 cm) with radius 2 cm (±.2 cm).

- Estimate the volume, with error bounds derived from differentials.
- Compare with exact error bound.

26.4
27.9