

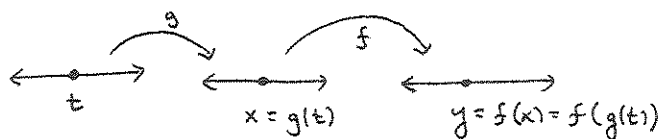
Math 2210-3
Wednesday March 3

§ 12.6 Chain Rule

Hw for Wed 3/10

12.6 1 (6, 12, 13, 17, 18) 19, (20) 29
12.7 1, (4, 6, 9, 15, 22)
12-8 1 (3, 7) 11, 12, (13, 15) 16 (19, 20, 22)
(24, 30, 31)

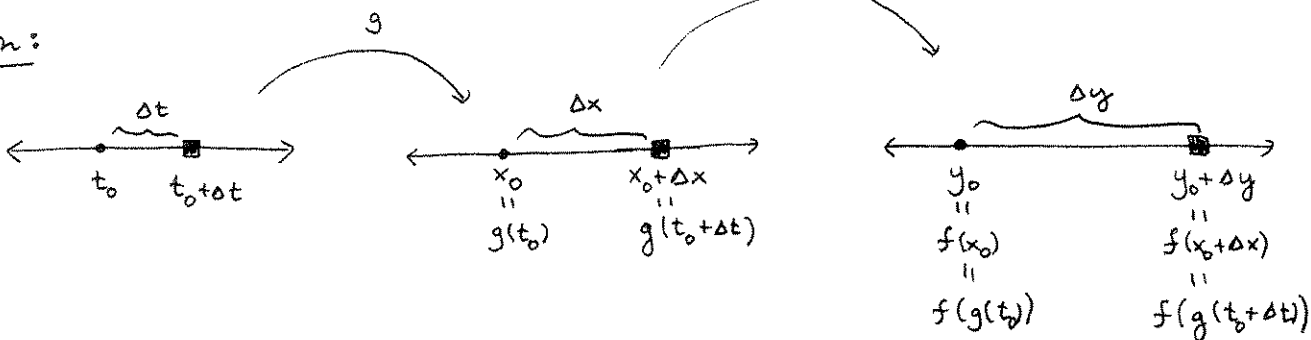
1210: functions of 1-variable



$$\frac{d}{dt} f(g(t)) = f'(g(t)) \cdot g'(t)$$

or, $\frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt}$

Reason:



Tangent approximation formula for f:

$$\Delta y = f'(x_0) \Delta x + (\Delta x) \varepsilon(\Delta x)$$

$$\frac{\Delta y}{\Delta t} = f'(x_0) \frac{\Delta x}{\Delta t} + \left(\frac{\Delta x}{\Delta t}\right) \varepsilon(\Delta x)$$

take $\lim_{\Delta t \rightarrow 0}$:

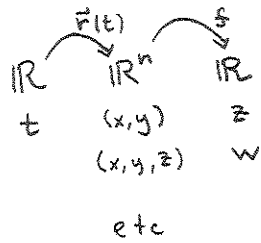
$$\frac{dy}{dt} = f'(g(t_0)) g'(t_0) + 0$$

$$\frac{\Delta x}{\Delta t} \varepsilon(\Delta x) \rightarrow g'(t_0) \cdot 0$$

since $\Delta x \rightarrow 0$
as $\Delta t \rightarrow 0$.

(2)

multivariable:



Chain rule

$$\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}'(t)$$

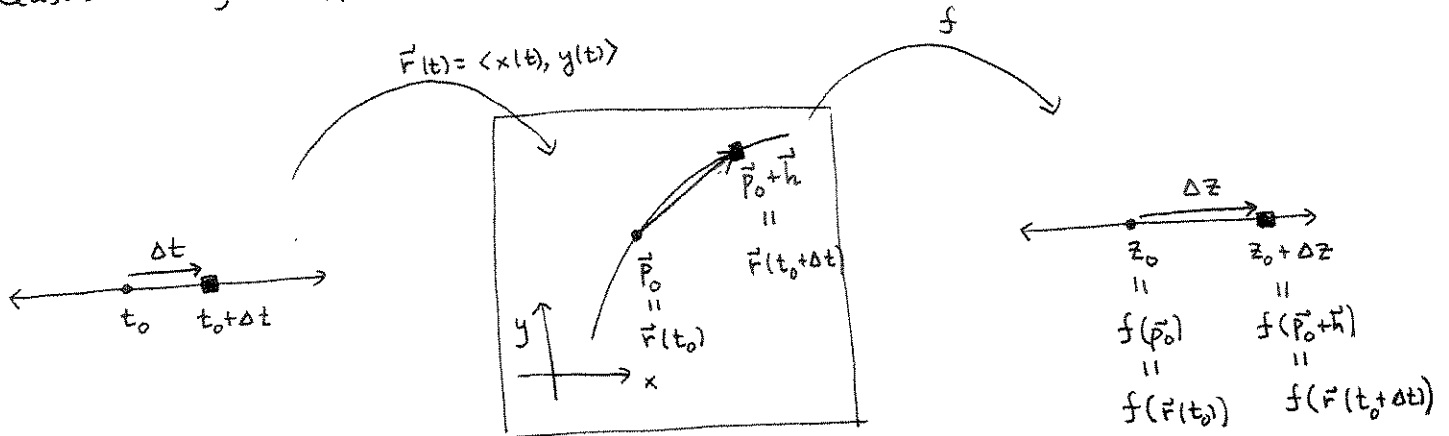
$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}$$

i.e.

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt}$$

etc.

Reason is tangent approximation formula!

affine approx for f

$$f(\vec{p}_0 + \vec{h}) = f(\vec{p}_0) + \nabla f(\vec{p}_0) \cdot \vec{h} + \vec{h} \cdot \vec{\epsilon}(\vec{h})$$

$$\Delta z = f(\vec{p}_0 + \vec{h}) - f(\vec{p}_0) = \nabla f(\vec{p}_0) \cdot \vec{h} + \vec{h} \cdot \vec{\epsilon}(\vec{h})$$

$$\frac{\Delta z}{\Delta t} = \nabla f(\vec{p}_0) \cdot \frac{\vec{h}}{\Delta t} + \frac{\vec{h}}{\Delta t} \cdot \vec{\epsilon}(\vec{h})$$

$$\text{Note } \frac{\vec{h}}{\Delta t} = \frac{\vec{r}(t_0 + \Delta t) - \vec{r}(t_0)}{\Delta t}$$

$$\rightarrow \vec{r}'(t_0) \text{ as } \Delta t \rightarrow 0$$

Let $\Delta t \rightarrow 0$:

$$(f \circ \vec{r})'(t_0) = \nabla f(\vec{r}(t_0)) \cdot \vec{r}'(t_0)$$

$$+ \vec{r}'(t_0) \cdot \vec{0}$$

because $\vec{h} \rightarrow 0$ as $\Delta t \rightarrow 0$
so $\vec{\epsilon}(\vec{h}) \rightarrow 0$ too.



Examples

$$z = x^3 y$$

$$x = 2t$$

$$y = t^2$$



← you can use "chains" like this to remember the multivariable chain rule

Find $\frac{dz}{dt}$

(a) by page 2 chain rule

(b) by writing z as a fun of t

Example (Related rates revisited)

A block of ice is melting in the sun,
remaining in the shape of a rectangular prism.

At noon the height z is 12" and is decreasing at a rate of 1"/hour.



depth y is 20", decreasing @ $\frac{1}{2}$ " / hour
length x is 30", decreasing @ $\frac{1}{2}$ " / hour.

How fast is volume changing?

$$V = xyz$$

chain rule for partial derivatives is identical: (really this is an application of what we've already done.)

④

i.e.
e.g.

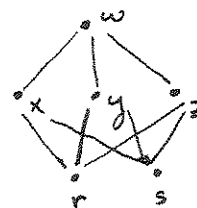
$$\begin{pmatrix} r \\ s \end{pmatrix} \rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} \rightarrow w$$

w is a fun of (x, y, z)

x, y, z are each funs of r, s .

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial s}$$



Example

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$z = x^2 + y^2$$

Find z_r
 z_θ

with the chain rule
and by direct substitution.

It takes practice to make the
chain rule feel natural!