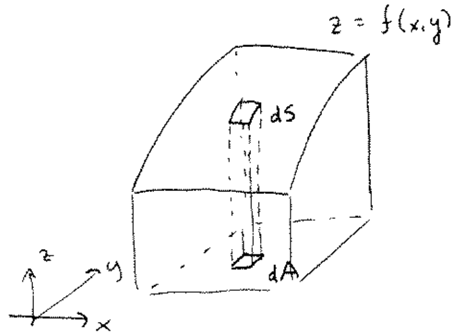
Math 2210-3
Monday March 29

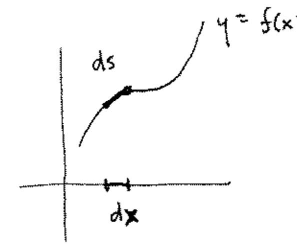
§13.6 Surface area of graphs and parametric surfaces

graphs:



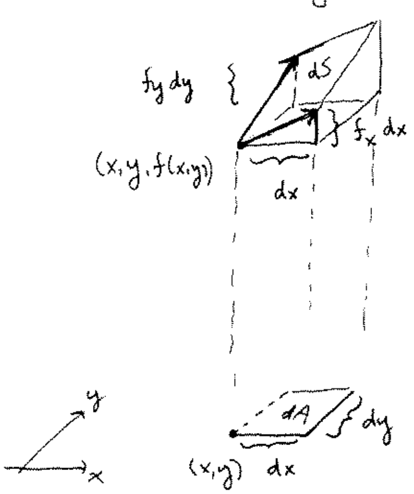
how is dS (the surface area element on the graph) related to dA?

1210 Calculus analog



$ds = \sqrt{1+f'(x)^2}\, dx.$

magnify:



$$dS = \| \langle dx, 0, f_x\, dx\rangle \times \langle 0, dy, f_y\, dy\rangle \|$$

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ dx & 0 & f_x\, dx \\ 0 & dy & f_y\, dy \end{vmatrix} = \langle -f_x, -f_y, 1\rangle\, dx\, dy$$

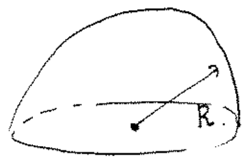
$$\boxed{dS = \sqrt{1+f_x^2+f_y^2}\; dA}$$

↑ accounts for the inclination of the graph

So, area of graph above D $\boxed{= \iint_D \sqrt{1+f_x^2+f_y^2}\; dA}$

Example 1:

Find the area of the upper ~~xxx~~ hemisphere of radius R.



$$x^2+y^2+z^2 = R^2$$

$$x^2+y^2+f(x,y)^2 = R^2$$

$$2x + 2ff_x = 0 \qquad f_x = \frac{-x}{f} = \frac{-x}{\sqrt{R^2-x^2-y^2}} \qquad f_x^2 = \frac{x^2}{R^2-x^2-y^2}$$

$$2y + 2ff_y = 0 \qquad f_y = \frac{-y}{f} = \frac{-y}{\sqrt{\quad}} \qquad f_y^2 = \frac{y^2}{R^2-x^2-y^2}$$

$$\sqrt{1+f_x^2+f_y^2} = \sqrt{1+\frac{x^2}{R^2-x^2-y^2}+\frac{y^2}{R^2-x^2-y^2}}$$

$$= \frac{R}{\sqrt{R^2-x^2-y^2}}$$

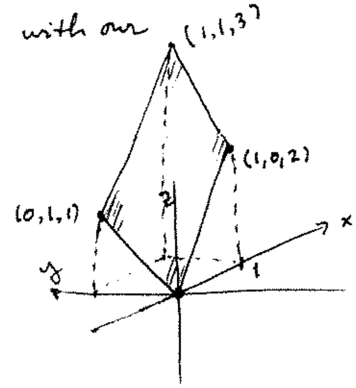
$$\text{Area} = \iint_{x^2+y^2\le R^2} \frac{R}{\sqrt{R^2-x^2-y^2}}\, dA$$

convert ~~to~~ polar!

$$= \int_0^{2\pi}\int_0^R \underbrace{\frac{R}{\sqrt{R^2-r^2}}\, r\,dr}_{-R(R^2-r^2)^{\frac{1}{2}}\Big]_0^R}\, d\theta$$

$$= \int_0^{2\pi} R^2\, d\theta = 2\pi R^2 \quad \checkmark$$

Exercise 2 Find the area of the part of the plane $z = 2x+y$ above the square $0 \le x \le 1$, $0 \le y \le 1$, using the surface area formula. Check with our old cross product formula



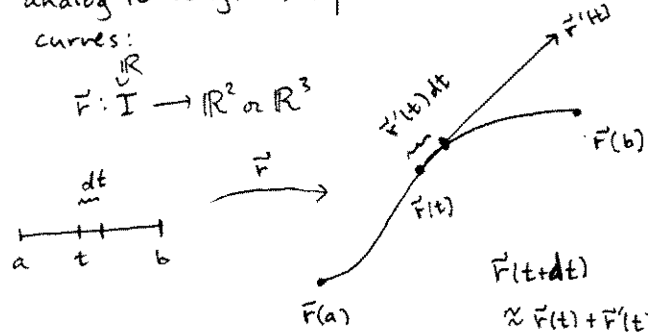
Area for <u>parametric surfaces</u> (14.5)

$$\vec{r} : \underset{\subset \mathbb{R}^2}{D} \longrightarrow \mathbb{R}^3$$

$$\vec{r}(u,v) = \begin{bmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{bmatrix}$$

analog to length for parametric curves:

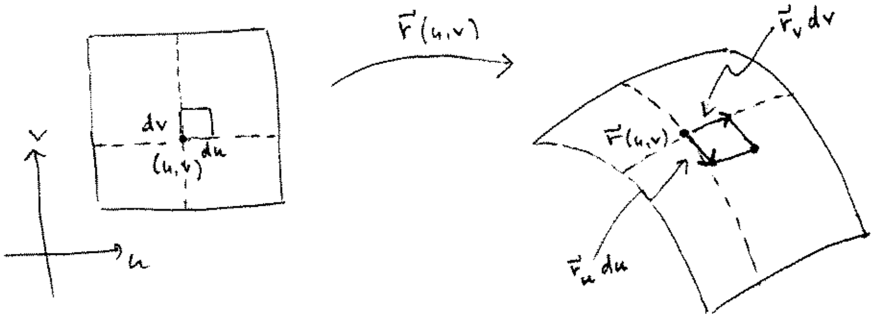
$$\vec{r} : \underset{\subset \mathbb{R}}{I} \longrightarrow \mathbb{R}^2 \text{ or } \mathbb{R}^3$$



$\vec{r}(t+dt) \approx \vec{r}(t) + \vec{r}\,'(t)\,dt.$

$ds = \|\vec{r}\,'(t)\|\,dt$

$L = \int_a^b \|\vec{r}\,'(t)\|\,dt$ Chapter 11.



$\vec{r}(u+du,v) \approx \vec{r}(u,v) + \vec{r}_u\,du$

$\vec{r}(u,v+dv) \approx \vec{r}(u,v) + \vec{r}_v\,dv$

$\vec{r}(u+du,v+dv) \approx \vec{r}(u,v) + \vec{r}_u\,du + \vec{r}_v\,dv$ affine approx formula

$dS = \|r_u\,du \times \vec{r}_v\,dv\| = \|\vec{r}_u \times \vec{r}_v\|\,du\,dv$ generalizes graph case, $\vec{r}(x,y) = \langle x, y, f(x,y)\rangle$

$$\text{Area} = \iint_D \|\vec{r}_u \times \vec{r}_v\|\,du\,dv$$

<u>Exercise 3</u> Check this gives the correct surface area for the cylindrical (piece) surface

$\vec{r}(\theta, z) = \langle 3\cos\theta, 3\sin\theta, z\rangle$

$0 \le \theta \le \pi$

$0 \le z \le 5.$