Math 2210-3
Wednesday March 17

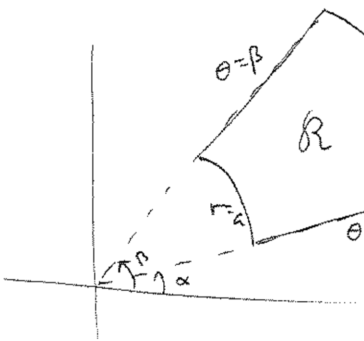
HW for after break: due March 31

13.3 1, 5, 11, 15, 16, 18, 21, 25, 27, 35, 36, 39
13.4 1, 7, 19, 20, 23, 27, 35
13.5 3, 11, 14 ← once you set up these integrals you may use software for ans.
13.6 1, 6, 7, 9

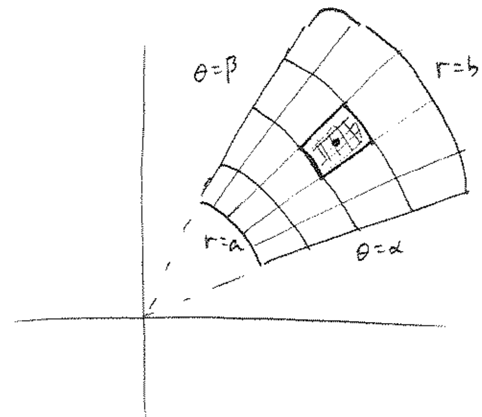
§13.4 Double integrals using polar coordinates

Consider a "polar rectangle"

$\alpha \le \theta \le \beta$
$a \le r \le b.$



partition this region using level curves of r and θ



We wish to compute

$$\boxed{\iint_R f(x,y)\,dA}$$

via a change of variables to polar coords

$x = r\cos\theta$
$y = r\sin\theta$

Riemann sum - sample at "midpoints"

$$\sum_{\substack{1\le i\le m\\ 1\le j\le n}} f(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*)\, r_i^* \,\Delta r_i \,\Delta\theta_j$$

This is a Riemann sum for

$$g(r,\theta) = f(r\cos\theta, r\sin\theta)\underbrace{r}_{\text{extra term!}}$$

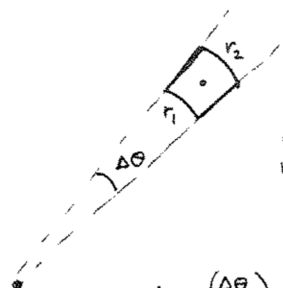
in a rectangle $a \le r \le b$, $\alpha \le \theta \le \beta$ of the r-θ plane.

As the partitioning size → 0, these Riemann sums converge to

$$\boxed{\int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta}$$

$$= \iint_R f(x,y)\,dA$$

Exact computation of area:



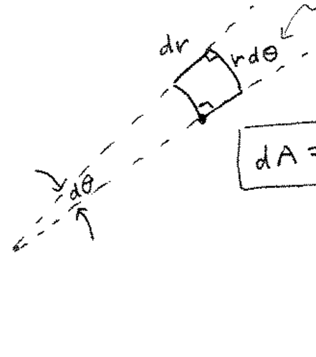
$$\Delta A = \underbrace{\left(\frac{\Delta\theta}{2\pi}\right)}_{\text{fraction of complete disk}}\left(\underbrace{\pi r_2^2}_{\text{radius } r_2\text{-disk area}} - \underbrace{\pi r_1^2}_{r_1 \text{ disk area}}\right)$$

$$= \frac{\Delta\theta}{\cancel{2\pi}}(r_2 - r_1)\frac{(r_2+r_1)}{2}$$

$$= (\Delta r)(\Delta\theta)\underbrace{\bar{r}}_{\text{average radius}}$$

Area picture with differentials (affine approx)

(an arc of radius r and angle dθ has length r dθ)



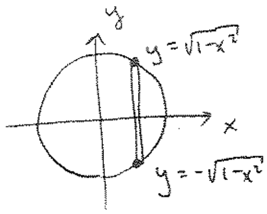
$$\boxed{dA = r\,dr\,d\theta}$$

Example: Verify that the area of the unit disk is π, using integration

(a)

old way

(we did this a few days ago.)



$$A = \iint_{\text{disk}} 1 \, dx\, dy$$

$$= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 1 \, dy\, dx = \int_{-1}^{1} 2\sqrt{1-x^2}\, dx$$

$$= 2\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x \right]_{-1}^{1}$$

(integral table #54)

$$= \sin^{-1} 1 - \sin^{-1}(-1)$$

$$= \frac{\pi}{2} - -\frac{\pi}{2} = \pi \quad !!$$

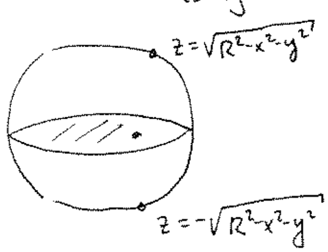
(b) polar coords C.O.V.

$$\iint_{\text{disk}} 1 \, dx\, dy = \int_0^{2\pi} \int_0^1 r\, dr\, d\theta \qquad \tfrac{1}{2}r^2\Big]_0^1$$

$$= \int_0^{2\pi} \frac{1}{2}\, d\theta$$

$$= \pi \quad !$$

Example Verify that the volume of a radius R ball is $\frac{4}{3}\pi R^3$, using a double integral.



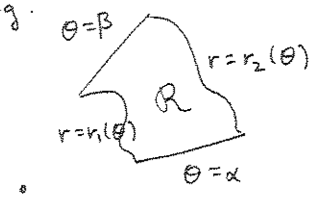
$$V = \iint_{x^2+y^2 \le R^2} 2\sqrt{R^2-x^2-y^2}\, dA$$

$x = r\cos\theta$
$y = r\sin\theta$

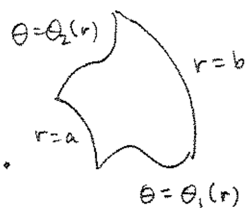
convert:

By the same reasoning as on page 1,
you can change to polar coords if
your region is radially or "angularly" simple

e.g.



$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_{r_1(\theta)}^{r_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$



$$\iint_R f(x,y)\,dA = \int_a^b \int_{\theta_1(r)}^{\theta_2(r)} f(r\cos\theta, r\sin\theta)\, r\, d\theta\, dr.$$

Example (p. 893)

Find the volume inside the
cylinder
$$x^2+(y-1)^2=1$$
$$(x^2+y^2=2y)$$
and below the paraboloid $z=x^2+y^2$
and above the x-y plane!

(ans $=\frac{3\pi}{2}$; use #113 integral table).