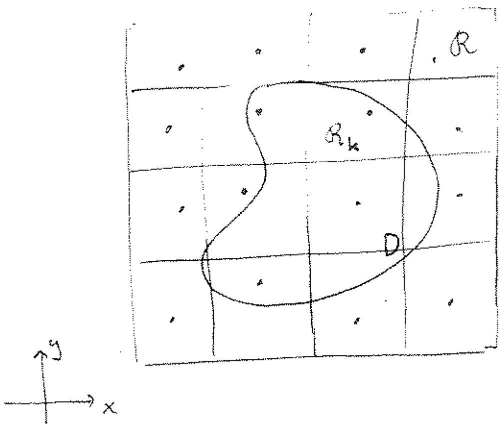
Math 2210-3
Monday March 15

§13.3 : Double integrals over non-rectangular domains

Not too hard to define: extend $f(x,y)$ to equal zero outside the domain, enclose domain in a big rectangle, and proceed as before!



Riemann sum: (as before)

$$\sum_k f(x_k^*, y_k^*)\,\Delta A_k$$

↑ this value will $= 0$ whenever (x_k^*, y_k^*) is not in D

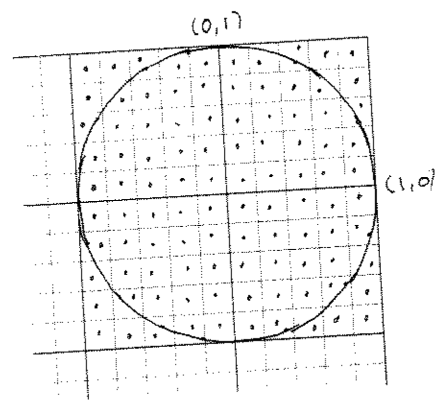
Theorem

If D is a "nice" domain (e.g. its boundary is a finite union of smooth curves)

and if f is continuous on D union its boundary,
then as $|P| \to 0$, the Riemann sums converge to a limit, called $\iint_D f(x,y)\,dA$.

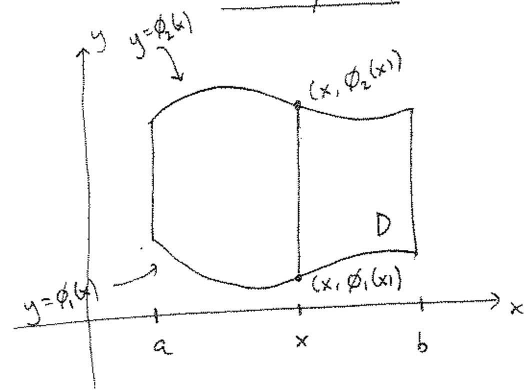
Example 1 Let D be the unit disk (radius 1), centered at the origin.

Estimate $\iint_D 1\,dA$



Using the partition indicated at right.
What geometric quantity are you estimating?
How many subrectangles are there?
How many of those give non-zero contributions to the Riemann sum?

Luckily, for "simple" domains we can use the method of slicing, and the interpretation of double integrals as representing volume, (or signed volume) to evaluate double integrals:

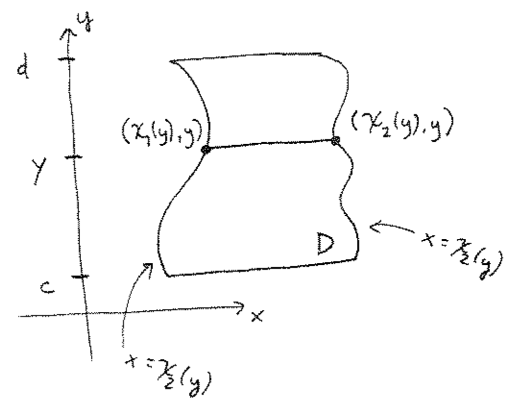


D is vertically simple (y-simple), if

$$D = \{(x,y) : a \le x \le b \text{ and } \phi_1(x) \le y \le \phi_2(x)\}$$

D is horizontally simple (x-simple), if

$$D = \{(x,y) : c \le y \le d \text{ and } \chi_1(y) \le x \le \chi_2(y)\}$$



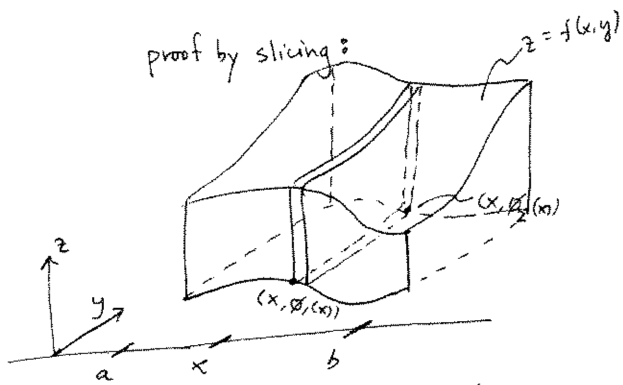
Theorem If D is y-simple

$$\iint_D f(x,y)\,dA = \int_a^b \left(\int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy \right) dx$$

Similarly, if D is x-simple,

$$\iint_D f(x,y)\,dA = \int_c^d \left(\int_{\chi_1(y)}^{\chi_2(y)} f(x,y)\,dx \right) dy.$$

proof by slicing:



$$\iint_D f(x,y)\,dA = V = \int_a^b dV$$

$$dV = A(x)\,dx = \left(\int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy \right) dx$$

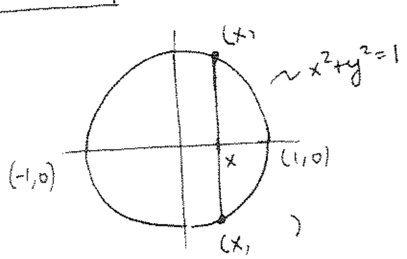
so $$\iint_D f(x,y)\,dA = \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy\,dx \quad ∎$$

Example 2: Let D be the unit disk from page 1.
Notice D is x-simple and y-simple.

Compute $\iint_D 1 \, dA$ using iterated integrals.

hint: Use # 54 in book cover integral table

picture helps:



Example 3 Compute the following iterated integral.
Then use the integral limits to sketch (identify) the domain being integrated over.

$$\int_0^1 \left(\int_{x^3}^{\sqrt{x}} x y^2 \, dy \right) dx$$

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> Int(Int(x*y^2,y=x^3..sqrt(x)),x=0..1);
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$$\int_0^1 \int_{x^3}^{\sqrt{x}} x\,y^2 \, dy \, dx$$

```
> int(int(x*y^2,y=x^3..sqrt(x)),x=0..1);
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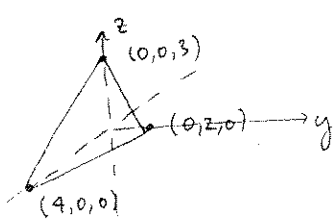
$$\frac{5}{}$$

Example 4 The domain for example 3 is y-simple and x-simple.
Re-evaluate the $\iint_D f(x,y)\,dA$ by expressing the iterated integral in ~~[illegible]~~ reverse order.

Example 5 Use a double integral to find the volume of the tetrahedron in the 1st octant, bounded by the 3-coordinate planes and the plane $3x + 6y + 4z = 12$



ans: $\frac{1}{3} Bh$

$= \frac{1}{3} \cdot 4 \cdot 3$

$= 4.$