Math 2210-3
Friday Jan 29 §11.5: vector-valued fcns, parametric curves.

Calculus for vector-valued functions (this was page 4 Wed.)

Consider

parametric curves with position vector

$\vec{r}(t) = \langle f(t), g(t) \rangle \longleftrightarrow$ in x-y plane

$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle \longleftrightarrow$ in $\mathbb{R}^3$

$$\boxed{\vec{r}\,'(t) := \lim_{\Delta t \to 0} \left(\frac{1}{\Delta t}\right)\underbrace{\left(\vec{r}(t+\Delta t) - \vec{r}(t)\right)}} \quad \boxed{\text{definition.}}$$

We call $\vec{r}\,'(t)$ the velocity vector (if t is time) or tangent vector

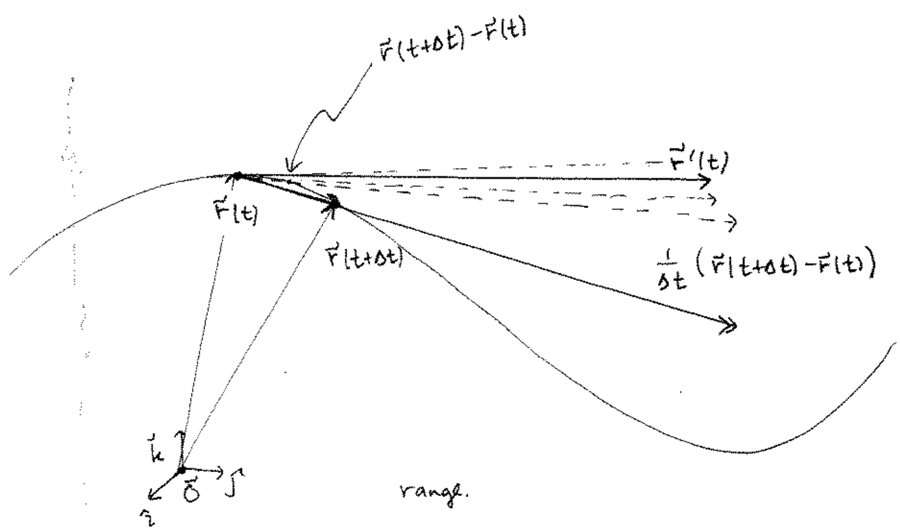
$$\frac{1}{\Delta t}\Big[f(t+\Delta t)\,\hat{\imath} + g(t+\Delta t)\,\hat{\jmath} + h(t+\Delta t)\,\hat{k} - \left(f(t)\,\hat{\imath} + g(t)\,\hat{\jmath} + h(t)\,\hat{k}\right)\Big]$$

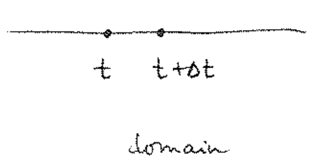
$$= \lim_{\Delta t \to 0} \left(\frac{f(t+\Delta t) - f(t)}{\Delta t}\right)\hat{\imath} + \left(\frac{g(t+\Delta t) - g(t)}{\Delta t}\right)\hat{\jmath} + \left(\frac{h(t+\Delta t) - h(t)}{\Delta t}\right)\hat{k}$$

(the limit of a vector expression is the vector which the expressions get arbitrarily close to as $\Delta t \to 0$, if it exists. this can be made precise with quantifiers "ε", "δ", see p. 579)

$$= f'(t)\,\hat{\imath} + g'(t)\,\hat{\jmath} + h'(t)\,\hat{k}$$

$$\boxed{\vec{r}\,'(t) = \langle f'(t), g'(t), h'(t) \rangle} \quad \boxed{\text{computation}}$$





$\boxed{\text{geometric meaning}}$

Example 1

For $\vec{r}(t) = \langle 2+t, 1+t^2 \rangle$

(which was page 1 Wednesday)

Compute $\dfrac{\vec{r}(t+\Delta t)-\vec{r}(t)}{\Delta t}$

at $t=-1$

for $\Delta t = .1$

$\Delta t = -.1$

Compute $\vec{r}\,'(t)$

$\vec{r}\,'(-1)$

Illustrate your computations

Math 2210 work

```
> restart:
> r:=t->[2+t,1+t^2];
  (r(-0.9)-r(-1))/.1;
  (r(-1.01)-r(-1))/(-.01);
  diff(r(t),t);
  subs(t=-1,diff(r(t),t));
```

$$r := t \to [2+t, 1+t^2]$$

$$[1., -1.9]$$

$$[1., -2.01]$$

$$[1, 2\,t]$$

$$[1, -2]$$

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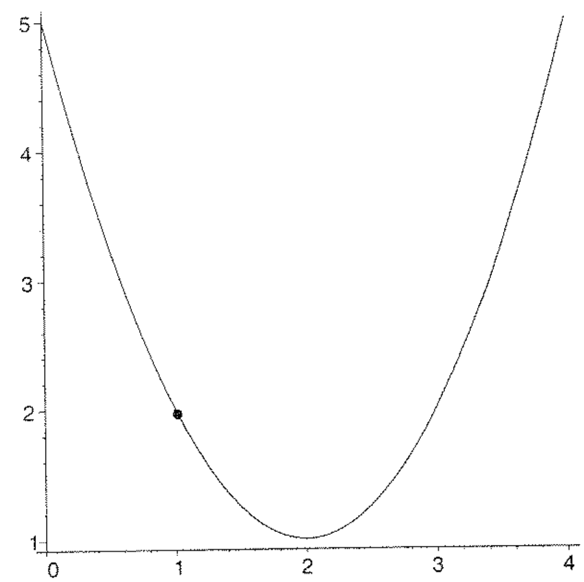
> with(plots):
Warning, the name changecoords has been redefined
> plot([2+t,1+t^2,t=-2..2],color=black,scaling=constrained);
```



Example 2 For a curve in the plane,

$\vec{r}(t) = \langle f(t), g(t) \rangle$, lying on a graph $y = F(x)$

can you draw a picture illustrating

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \text{ ?}$$

(We used Calculus chain rule to see this another way on Wednesday.)

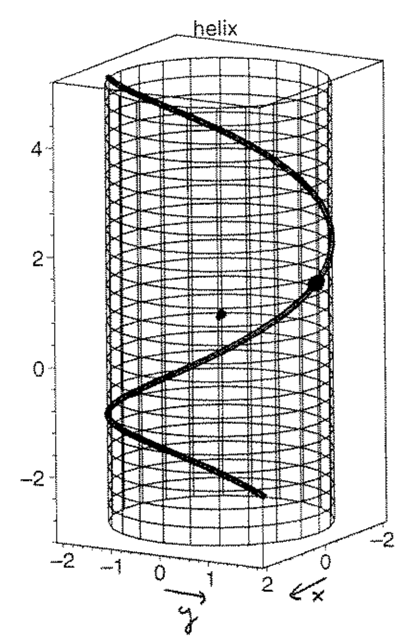
Example 3

Consider the helix

$$\vec{r}(t) = \langle \underbrace{2\cos t, 2\sin t}_{\text{proj to x-y plane is going around circle radius 2}}, \underbrace{t}_{\text{increasing at const rate in z-dir.}} \rangle$$

plot $\vec{r}(\pi/2)$ as a position vector from the origin

$\vec{r}\,'(\pi/2)$ as a tangent vector to the helix.



Physics: If t = time

$\dfrac{\vec{r}(t+\Delta t)-\vec{r}(t)}{\Delta t}$ = average velocity   net displacement per unit time

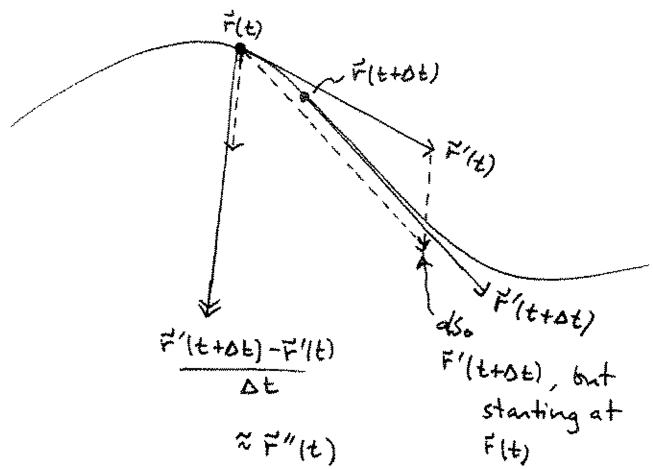
$\left\| \dfrac{\vec{r}(t+\Delta t)-\vec{r}(t)}{\Delta t} \right\|$ = average speed $\dfrac{\text{net distance}}{\text{time taken}}$

$\vec{r}\,'(t)$ = velocity $\vec{v}(t)$

$\|\vec{r}\,'(t)\|$ = speed $v(t)$

$$\boxed{\frac{d\vec{v}}{dt} = \vec{r}\,''(t) = \underset{\vec{a}(t)}{\text{acceleration}}}$$ Measures rate of change of velocity!

Example 4) Compute the acceleration vector for examples 1, 3, at the indicated t-values
Add to pictures



Curve length: $\vec{r}:[a,b] \to \mathbb{R}^n$

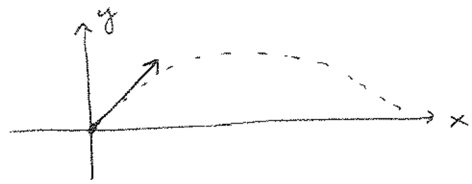
$$L = \int_a^b \|\vec{r}\,'(t)\|\,dt$$

- see math derivation §5.4 page 296 for plane curve
- makes sense since $\underbrace{\underbrace{\|\vec{r}\,'(t)\|}_{\text{speed}}\ \underbrace{dt}_{\text{time}}}_{\text{distance traveled in time } dt}$

Example 5) Find the length of one loop of the helix, $\vec{r}(t) = \langle 2\cos t, 2\sin t, t\rangle$
$0 \le t \le 2\pi$
in example 3)   This is a redo from week 1!

~~The~~
Antidifferentiation is important too!

Example 6 A projectile is thrown with an initial speed of v_0 m/s at an angle θ radians with the horizontal. Assuming only gravitational acceleration, describe its motion



$\vec{a} = \langle 0, -g \rangle = \vec{r}\,''(t).$

$\vec{v}_0 = v_0 \langle \cos\theta, \sin\theta \rangle$

To proceed, we will need to use differentiation rules for vector-valued functions. These follow from differentiation rules for scalar functions

1. $D_t\left(\vec{F}(t) + \vec{G}(t)\right) = \vec{F}\,'(t) + \vec{G}\,'(t)$
2. $D_t\left(c\vec{F}(t)\right) = c\vec{F}\,'(t)$
3. $D_t\left(h(t)\vec{F}(t)\right) = h'F + hF'$
4. $D_t\left(\vec{F}(t)\cdot\vec{G}(t)\right) = \vec{F}\,'\cdot\vec{G} + \vec{F}\cdot\vec{G}\,'$
5. $D_t\left(\vec{F}(t)\times\vec{G}(t)\right) = \vec{F}\,'\times\vec{G} + \vec{F}\times\vec{G}\,'$

(3–5: product rules)

6. $D_t\left(\vec{F}(g(t))\right) = \vec{F}\,'(g(t))\,g'(t)$ } Chain rule. $g(t)$ a scalar function

Discuss!